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Singularities in $U(2)$-invariant 4d Ricci flow

by

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Abstract

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Firstly, we analyze the steady Ricci soliton equation for a certain class of metrics on complex line bundles over Kähler-Einstein manifolds of positive scalar curvature. We show that these spaces admit a non-collapsed steady gradient Ricci soliton metric. In the four (real-) dimensional case, this yields a new family of non-collapsed steady Ricci solitons on the complex line bundles $O(-k)$, $k \geq 3$, over $\mathbb{C}P^1$. These solitons are $U(2)$-invariant, non-Kähler, and asymptotic to the the quotient of the four dimensional Bryant soliton by $\mathbb{Z}_k$. As a byproduct of our work we also find Taub-Nut like Ricci solitons on $\mathbb{R}^4$ and demonstrate a new proof for the existence of the Bryant soliton.

Secondly, we investigate the formation of singularities in four dimensional $U(2)$-invariant Ricci flow and show that the Eguchi-Hanson space can occur as a blow-up limit. In particular, we prove that starting from a class of asymptotically cylindrical $U(2)$-invariant initial metrics on TS^2, a Type II singularity modeled on the Eguchi-Hanson space develops in finite time and the only possible blow-up limits are (i) the Eguchi-Hanson space, (ii) the flat $\mathbb{R}^4/\mathbb{Z}_2$ orbifold, (iii) the 4d Bryant soliton quotiented by $\mathbb{Z}_2$, and (iv) the shrinking cylinder $\mathbb{R} \times \mathbb{R}P^3$. It also follows from our work that in four dimensional Ricci flow an embedded two dimensional sphere of any self-intersection number $k \in \mathbb{Z}$ may collapse to a point in finite time and thereby produce a singularity. For $|k| \geq 3$ the singularities we construct are of Type II, yielding a new infinite family of Type II singularities. Numerical simulations indicate that their blow-up limits are the four dimensional steady Ricci solitons described above.

Contents

List of Figures

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Chapter 1

A family of non-collapsed steady Ricci solitons

1.1 Introduction

In this chapter we construct new families of non-collapsed and non-Kähler steady Ricci solitons in 4d and higher dimensions. A Ricci soliton (M, g) is a self-similar solution to the Ricci flow equations

$$\partial_t g_{ij} = -2Ric_{ij} \tag{1.1.1}$$

that, up to diffeomorphism, homothetically shrinks, expands, or remains steady under Ricci flow. We will study only steady gradient solitons which satisfy the equation

$$Ric_{ij} + \nabla_i \nabla_j f = 0, \tag{1.1.2}$$

for a smooth potential function $f : M \to \mathbb{R}$. Solitons are interesting objects, because they are candidates for blow-up limits of singularities in Ricci Flow. In particular, Type I singularities correspond to shrinking solitons and all Type II singularities known so far are modelled on steady solitons. The new non-collapsed steady solitons we find here are likely to occur as singularity models in Ricci flow. In paper [AW19], still in preparation, we have conducted numerical simulations verifying this.

In three dimensions the classification of non-expanding solitons has largely been carried out and singularity formation is well-understood. But in four dimensions, even though finding non-expanding Ricci solitons is a fundamental problem, to date there are surprisingly few examples known. The last one was discovered by Feldman, Ilmanen, and Knopf [FIK03] — the FIK shrinker — also shown to occur as a singularity model by Maximo [M14]. Before this, Cao [Cao96] had constructed a $U(2)$-invariant steady Kähler-Ricci soliton. This soliton, however, is collapsed and hence, as shown in Perelman's work [Perl02], does not appear as a blow-up limit. The rotationally symmetric Bryant soliton [B05] is the last non-collapsed, non-Kähler, non-expanding soliton discovered in 4d.

Our new solitons in 4d are asymptotic to the 4d Bryant soliton's quotient by a cyclic group $\mathbb{Z}_k$ of order $k \geq 3$. But the underlying topology is different, because our solitons exist on the completion of the space $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_k$, $k \geq 3$, obtained by adding an S^2 at the origin. Relying on an idea of Page and Pope [PP87], our methods carry over to complex line bundles over Kähler-Einstein manifolds of positive scalar curvature. This allows us to prove the existence of non-collapsed steady solitons on such bundles, if their degrees are sufficiently large. In doing so we obtain $(2n+2)$-dimensional solitons on the line bundles $O(k)$, $k \geq n+1$, of $\mathbb{C}P^n$, which are also asymptotic to a quotient of the $(2n+2)$-dimensional Bryant soliton.

In this chapter we consider metrics for which the Ricci soliton equations reduce to a system of ODE's. In the lead up to the construction of the non-collapsed solitons, we show that a 1-parameter family of complete solutions exists. These solutions, however, correspond to collapsed solitons. [Wink17] and [Stol17] independently and by different methods discovered those collapsed solitons. Our main result, on the other hand, is showing that a critical choice of parameter for the initial condition of the ODE yields a non-collapsed soliton. In the final part of Chapter 1 we apply our methods to the spaces $\mathbb{R}_{>0} \times S^n$, $n \geq 2$, completed by a point. Thereby we prove the existence of a new 1-parameter family of Taub-Nut like Ricci solitons. Furthermore we demonstrate an alternative proof of the Bryant soliton's existence in all dimensions ≥ 3.

Below we would like to further motivate why the study of Ricci solitons is important and describe the geometries of our solitons in more detail. Solitons are important objects in the study of Ricci flow because they arise as blow-up limits of singularities: Let $g(t)$, $t \in [0, T)$ be a smooth solution to the Ricci flow on a closed manifold M with $T < \infty$. Suppose that the Ricci flow develops a singularity at time T, i.e. there exists a point $p \in M$ and a sequence of times $t_i \to T$ such that the curvatures $K_i = |Rm_{g(t_i)}|(p)$ at p tend to infinity as $t_i \to T$. Then by Perelman's work it follows that the sequence of dilated solutions $(M, g_i(t))$

$$g_i(t) := K_i g\left(t_i + \frac{t}{K_i}\right), \quad t \in [-K_i t_i, 0], \tag{1.1.3}$$

converges in a suitable sense to a complete ancient solution $(M_\infty, g_\infty(t))$ of the Ricci Flow [ChI, Theorem 6.68], which is referred to as the singularity model. A solution to the Ricci flow is ancient if it is defined for times $\infty < t < T_0$ where $T_0 \in \mathbb{R} \cup \{\infty\}$. Note that the topology of M_∞ can be very different from M.

Hamilton [Ham95, Section 16] distinguishes between Type I and Type II finite time singularities, which are defined by the rate at which the curvature tends to infinity at a singularity. For a Type I singularity the curvature blows up at a rate $\sup_i (T - t_i) K_i < \infty$ and for a Type II singularity at a rate $\sup_i (T - t_i) K_i = \infty$. The blowup limit of a Type I singularity is a shrinking soliton [N10], [EMT11]. For Type II singularities, on the other hand, it is not known whether the blow-up limit must be a soliton. However, the only known Type II singularities so far are all modeled on the Bryant soliton [GZ08], [AIK11], [W14]. We would also like to point out that not all solitons arise as blow up limits. Due to Perelman's no local collapsing theorem [Perl02, Section 4], only non-collapsed solitons can emerge as blow-ups. (M, g) is said to be κ-non-collapsed below the scale $r > 0$ at the point x if $|Rm(g)| \leq r^{-2}$ for all $y \in B(x, r)$ and

$$\frac{\mathrm{Vol} B(x, r)}{r^n} \geq \kappa. \tag{1.1.4}$$

A soliton is non-collapsed if for some $\kappa > 0$ it is κ-non-collapsed at all scales and points.

In this chapter we will construct non-collapsed steady gradient solitons on manifolds that are the total space of certain complex line bundles. In four dimensions the topology and geometry of these manifolds are easy to describe. They are warped product metrics

$$g = ds^2 + g_{a(s),b(s)} \tag{1.1.5}$$

on $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_k$, $k \in \mathbb{N}$, that are completed by adding an S^2 at $s = 0$. Here s is the parametrization of the $\mathbb{R}_{\geq 0}$ factor. In the language of complex geometry these spaces are diffeomorphic to the blowup of $\mathbb{C}^2/\mathbb{Z}_k$ at the origin. To describe the metric $g_{a(s),b(s)}$ on the cross-sectional $S^3/\mathbb{Z}_k$, recall the Hopf fibration $\pi : S^3 \to S^2$ arising from the multiplicative action of

$$S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\} \subset \mathbb{C} \tag{1.1.6}$$

on

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2. \tag{1.1.7}$$

If we equip S^3 and S^2 with the standard round metrics of curvatures 1 and 4 respectively, S^1 acts by isometries and π is a Riemannian submersion. This is not the only metric with this property. We can define squashed metrics $g_{a,b}$ on S^3 by rescaling the vertical S^1-fiber directions by a factor of a and the horizontal directions by a factor of b relative to the standard round metric. The cross-sectional metric $g_{a(s),b(s)}$ is defined by taking the quotient of S^3 by

$$\mathbb{Z}_k = \{e^{2\pi i \frac{l}{k}} \mid l = 0, 1, \cdots, k-1\} \subset \mathbb{C} \tag{1.1.8}$$

and letting a and b vary with s. Therefore we can write the metric g as

$$g = ds^2 + a(s)^2 \sigma \otimes \sigma + b(s)^2 \pi^* g_{S^2(\frac{1}{2})}, \tag{1.1.9}$$

where σ is dual to the vertical vector field on S^3 obtained by the S^1 action and $g_{S^2(\frac{1}{2})}$ is the standard round metric on S^2 of curvature 4. We will show that for a metric of this form the soliton equation (1.1.2) reduces to a system of ODEs for a, b and f.

We complete the metric by taking $a(0) = 0$ and $b(0) > 0$, i.e. shrinking the S^1 fibers of the cross-sectional $S^3/\mathbb{Z}_k$ to zero at $s = 0$. Thus at $s = 0$ we are left with an S^2, showing that the manifold can be thought of as a radially filled in S^1 bundle over S^2 or, since S^2 is a complex manifold, as a complex line bundle over S^2. In the language of complex geometry these complex line bundles are the bundles $O(k)$ over complex projective space $\mathbb{C}P^1$. As the S^1 fibers of $S^3/\mathbb{Z}_k$ are parametrized by $0 \leq \theta < \frac{2\pi}{k}$ and the circumferences $\frac{2\pi}{k} a(s)$ of the S^1 fibers behave like $\frac{2\pi}{k} a'(0) s + O(s^2)$ as $s \to 0$, we must require $a'(0) = k$, which is a neccessary condition to obtain a smooth metric at $s = 0$. This is how the topology of the manifold enters the analysis of solving the Ricci soliton equation.

In four dimensions it will follow from our main theorem 1.1.2 stated below that

Corollary 1.1.1 (Corollary of theorem 1.1.2)**.** *On the completion of the warped product metric $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_k$, or in the language of complex geometry, on the complex line bundle $O(k)$ over $\mathbb{C}P^1$ there exists a complete non-collapsed steady Ricci soliton when $k > 2$.*

We will also prove that the asymptotics of these solitons are $a \sim b \sim C\sqrt{s}$ as $s \to \infty$ for $C > 0$ a constant, showing that they are asymptotic to the quotient of the 4d Bryant soliton [B05] by $\mathbb{Z}_k$. The above result generalizes directly to higher dimensional warped product metrics of the form

$$g = ds^2 + g_{a(s),b(s)} = ds^2 + a(s)^2\sigma \otimes \sigma + b(s)^2\pi^* g_{\mathbb{C}P^n} \tag{1.1.10}$$

on $\mathbb{R}_{>0} \times S^{2n+1}/\mathbb{Z}_k$, $n, k \in \mathbb{N}$. Here the metric $g_{a(s),b(s)}$ is defined analogously via the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ and $g_{\mathbb{C}P^n}$ is the Fubini-Study metric. We will show that when $k > n + 1$ there exists a non-collapsed steady gradient soliton on these spaces.

Many important metrics are of the form (1.1.10). For example in the case $k = n + 1$ the Ricci flat Eguchi-Hanson metric [EH79] and its higher dimensional generalizations [Cal79], [FG81] [AG03], or when $k < n + 1$ the Ricci flat Taub-Bolt metrics [P78] and the shrinking Khler-Ricci solitons found by Feldman, Ilmanen and Knopf [FIK03].

For the metrics above the Ricci soliton equation (1.1.2) reduces to a system of ODEs in a, b and f. Relying on ideas developed in [BB85] and [PP87], we obtain the same system of ODEs for certain metrics on complex line bundles over Khler-Einstein manifolds of positive scalar curvature and our methods carry over. We will describe the setup here: Let M denote the total space of a complex line bundle over a Khler-Einstein manifold $(\hat{M}, J, \hat{g})$ of positive scalar curvature. Let ω and ρ be the Khler and Ricci forms respectively and assume that the metric $\hat{g}$ is scaled such that $\rho = 2(n+1)\omega$. $\frac{\rho}{2\pi} \in H^2(\hat{M}, \mathbb{Z})$ is the Chern class of the canonical bundle of $\hat{M}$ and therefore integral. Thus we can write $\frac{\rho}{2\pi} = p\sigma$, for $p = p(\hat{M}, \omega) \in \mathbb{N}$ and $\sigma \in H^2(\hat{M}, \mathbb{Z})$ an indivisible cohomology class. We will be studying the complex line bundles whose Chern class is equal to $k\sigma$ for $k \in \mathbb{N}$. In the following we will denote such a line bundle by L_k omitting the dependence on $(\hat{M}, J, \hat{g})$ and consider metrics that locally are of the form

$$g = ds^2 + a(s)^2 (d\tau - 2A)^2 + b(s)^2\hat{g}, \tag{1.1.11}$$

where A is a connection 1-form satisfying $dA = \omega$ on $\hat{M}$, $\tau \in [0, 2\pi)$ is an angular coordinate of the S^1 subbundle of L_k and s is the radial coordinate. For $\mathbb{C}P^n$ equipped with the Fubini-Study metric we have $p = n + 1$ and $L_k = O(k)$. Note also that the above warped product metrics (1.1.9) can be written in the form (1.1.11), where the functions $a(s)$ and $b(s)$ have the same geometrical interpretation and the 1-form $d\tau - 2A$ corresponds to σ. The main result of this chapter is

Theorem 1.1.2. *Let $(\hat{M}, J, \hat{g})$ be a Khler-Einstein manifold of positive scalar curvature. Then there exists a non-collapsed steady gradient Ricci soliton on L_k when $k > p(\hat{M}, \omega)$. The asymptotics of these solitons are $a \sim b \sim C\sqrt{s}$ as $s \to \infty$ for $C > 0$ a constant.*

In the final part of this chapter we construct Taub-Nut like Ricci solitons on $\mathbb{R}^{2n+2}$ and give a new proof for the existence of the Bryant soliton in dimensions ≥ 3. These results will follow with relative ease from the methods developed in the first part of the chapter. This is because in the case $k = 1$, the warped product metric (1.1.10) can be interpreted as a metric on the completion of $\mathbb{R}_{s>0} \times S^{2n+1}$ by a point instead of an S^2. The soliton equations remain unchanged and only the initial conditions need to be modified to account for the change in topology, i.e. we will need to require $a = b = 0$ and $a' = b' = 1$ at $s = 0$ to ensure smoothness of the metric at the origin. The Taub-Nut metrics [T51], [H77], [BB85] are all of this form. Notice also that when $a = b$ everywhere we obtain a rotationally symmetric metric on $\mathbb{R}^{2n+2}$. This allows us to apply our methods to give another existence proof of the Bryant soliton in even dimensions ≥ 4. Surprisingly, the existence proof also carries over to odd dimensions, because the analytical structure of the equations remains unchanged. We merely lose the geometrical interpretation of a and b.

1.2 Gradient steady Ricci soliton equations

In the appendix A we show how the steady gradient Ricci soliton equation (1.1.2) reduces to the following system of ODEs for a metric of the form (1.1.11)

$$f'' = \frac{a''}{a} + 2n\frac{b''}{b} \tag{1.2.1}$$

$$a'' = 2n\left(\frac{a^3}{b^4} - \frac{a'b'}{b}\right) + a'f' \tag{1.2.2}$$

$$b'' = \frac{2n+2}{b} - 2\frac{a^2}{b^3} - \frac{a'b'}{a} - (2n-1)\frac{(b')^2}{b} + b'f', \tag{1.2.3}$$

where $(f, a, b) : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ are functions depending on s only and n is the complex dimension of the base manifold $\hat{M}$. Note that these equations are also the soliton equations for the metrics (1.1.9) and (1.1.10), because they are special cases of the general metric (1.1.11).

The boundary conditions imposed on a, b and f to ensure smoothness of the metric at $s = 0$ depend on the topology of the underlying manifold through p and k. In particular τ has a period of $\Delta\tau = \frac{2\pi p}{(n+1)k}$, which follows by either considering the holonomy of the connection A or the construction of the line bundle given the Chern class $k\sigma$. Therefore we must require $a'(0)\Delta\tau = 2\pi$ for the metric not to have a conical singularity at $s = 0$. Furthermore taking a to be smoothly extendable to an odd function and b, f to be smoothly extendable to even functions around $s = 0$ we can ensure smoothness of the metric and f at $s = 0$. Notice also that the equations $(1.2.1) - (1.2.3)$ only depend on f' and f'', so that we can assume without loss of generality that $f(0) = 0$. Finally by the scaling symmetry

$g \to \alpha g$, $\alpha \in \mathbb{R}$ we can fix $b(0) = 1$. In summary our boundary conditions at $s = 0$ therefore read

$$a = 0 \qquad a' = (n+1)\frac{k}{p} \tag{1.2.4}$$

$$b = 1 \qquad b' = 0 \tag{1.2.5}$$

$$f = 0 \qquad f' = 0. \tag{1.2.6}$$

For sections 2-7 of this chapter we will implicitely assume these boundary conditions in all lemmas and theorems stated. Note that in the case that the underlying manifold is the complex line bundle $O(k)$ on $\mathbb{C}P^n$, we have $p = n+1$ and therefore $a'(0) = k$.

The equations (1.2.1) – (1.2.3) with above boundary conditions are degenerate at $s = 0$ and we must specify $f''(0)$ to obtain a unique solution. This is further explained in appendix B, where we prove the following theorem:

Theorem 1.2.1. *Fix $n \in \mathbb{N}$ and $a_0, f_0^* \in \mathbb{R}$. Then there exists an $\epsilon > 0$ such that*

1. *For any $|f_0 - f_0^*| < \epsilon$ there exists a unique analytic solution $(f, a, b) : (-\epsilon, \epsilon) \setminus \{0\} \to \mathbb{R}^3$ to the soliton equations (1.2.1)-(1.2.3) satisfying the initial conditions $a(0) = 0$, $a'(0) = a_0$, $b(0) = 1$, $b'(0) = 0$, $f(0) = f'(0) = 0$ and $f''(0) = f_0$.*
2. *a is an odd function and b, f are even functions*
3. *The solution (f, a, b) depends analytically on f_0*

By this theorem and standard results in the theory of ordinary differential equations it follows that any solution $(f, a, b) : 0 \in I \to \mathbb{R}^3$ of (1.2.1) – (1.2.3) depends smoothly on $f''(0)$.

1.3 Evolution equations for $Q = \frac{a}{b}$, f and R

From the soliton equations (1.2.1)-(1.2.3) we can compute that the quotient $Q = \frac{a}{b}$ satisfies the following ODE

$$Q'' = \left(f' - (2n+1)\frac{b'}{b}\right)Q' + \frac{2n+2}{b^2}\left(Q^3 - Q\right), \tag{1.3.1}$$

from which we easily obtain the following key lemma:

Lemma 1.3.1. *Let $(f, a, b) : I \to \mathbb{R}^3$ be a solution to the soliton equations (1.2.1)-(1.2.3). Assume that $s^* \in I$ is a critical point of Q. If $Q(s^*) > 1$ ($0 < Q(s^*) < 1$) then Q has a strict local minimum (maximum) at s^*.*

Proof. Follows from (1.3.1). □

Remark 1.3.2. From lemma 1.3.1 and the boundary condition $Q(0) = 0$ it follows that if $Q > 1$ at some $s_0 > 0$ we must have that $Q' > 0$ for $s > s_0$.

Applying the Bianchi identity to the soliton equation we can obtain the following lemma.

Lemma 1.3.3. *For a steady gradient soliton $R_{ij} + \nabla_i \nabla_j f = 0$ the identity*

$$\nabla_k R_j^{\ k} = R_j^{\ k} \nabla_k f = \frac{1}{2} \nabla_j R \tag{1.3.2}$$

holds true.

Proof. Using the contracted Bianchi identity $\nabla_a R^a_e = \frac{1}{2} \nabla_e R$ we can compute

$$\begin{aligned}
\nabla_k R^k_j &= -\nabla_k \nabla^k \nabla_j f \\
&= (\nabla_j \nabla_k - \nabla_k \nabla_j) \nabla^k f - \nabla_j \nabla_k \nabla^k f \\
&= R^k_{\ ajk} \nabla^a f + \nabla_j R \\
&= -R_{aj} \nabla^a f + 2 \nabla_k R^k_j
\end{aligned}$$

Thereby we obtain the desired result. □

This result allows us to obtain nice evolution equations for the potential function f and the scalar curvature R.

Lemma 1.3.4. *The potential function f of a steady gradient soliton satisfies*

$$\Delta f - |\nabla f|^2 = -R(0) = 2f''(0) \tag{1.3.3}$$

Proof. Making use of identity (1.3.2), a computation shows that $\nabla_i \left(\Delta f - |\nabla f|^2\right) = 0$, from which the first equality of (1.3.3) follows. To prove the second equality, note that we can eliminate the second derivatives in the expression (1.10.5) for the scalar curvature R with help of the soliton equations (1.2.2)-(1.2.3). We thus obtain that

$$R = 2n\frac{a^2}{b^4} - \frac{2n(2n+2)}{b^2} + 4n\frac{a'b'}{ab} + 2n(2n-1)\left(\frac{b'}{b}\right)^2 - 2f'\left(\frac{a'}{a} + 2n\frac{b'}{b}\right). \tag{1.3.4}$$

Applying L'Hôpital's rule and noting that $b''(0) = n + 1$ by (1.2.3), we then deduce that $R(0) = -2f''(0)$. □

It is a well-known fact that $R \geq 0$ for any complete ancient solution to Ricci flow(see for instance [Chen09, Corollary 2.5]). Thus the second equality in (1.3.3) implies that we must require $f''(0) \leq 0$. In the rest of the chapter we will assume this.

We may similarly derive an evolution equation for R

Lemma 1.3.5. *The scalar curvature of a gradient steady soliton satisfies*

$$\Delta R + 2|Ric|^2 = \nabla^i R \nabla_i f \tag{1.3.5}$$

Proof. Applying the Bianchi identity (1.3.2) we obtain

$$\nabla^j \nabla_j R = 2\left(\nabla^j R_j^k\right) \nabla_k f + 2R_j^k \nabla_k \nabla^j f, \tag{1.3.6}$$

from which the desired result follows. □

1.4 Monotonicity properties of a, b, f, f' and R

Using the soliton equations (1.2.1)-(1.2.3) and evolution equations for f and R derived in the section above, we deduce various monotonicity properties of a, b, f, f' and R for $Q < \sqrt{n+1}$.

Lemma 1.4.1. *Let $s_0 > 0$ and $(f, a, b) : [0, s_0) \to \mathbb{R}^3$ be a solution to the soliton equations (1.2.1)-(1.2.3). Then a is strictly increasing on $[0, s_0)$ and b is strictly increasing on any interval $0 \in I' \subset [0, s_0)$ on which $Q < \sqrt{n+1}$. Furthermore b' changes its sign at most once on the interval $[0, s_0)$.*

Proof. Whenever $a' = 0$, we have $a'' = 2n\frac{a^3}{b^4}$. Since $a'(0) > 0$ the monotonicity of a follows. Similarly, whenever $b' = 0$, we have $b'' = 2\frac{n+1-Q^2}{b}$. By applying L'Hôpital's rule to (1.2.3) we compute that $b''(0) = n + 1 > 0$. This in conjunction with the boundary condition $b'(0) = 0$ implies the monotonicity of b when $Q < \sqrt{n+1}$. Therefore b' can change sign only when $Q^2 \geq n+1$. Since Q is strictly increasing when $Q > 1$ and $b'' = 2\frac{n+1-Q^2}{b}$ whenever $b' = 0$, it follows that b' changes its sign at most once. □

We can prove the following lemma in a similar fashion

Lemma 1.4.2. *Let $s_0 \in \mathbb{R}_{>0}$ and $(f, a, b) : [0, s_0) \to \mathbb{R}^3$ be a solution to the soliton equations (1.2.1)-(1.2.3). The following holds true*

1. *if $f''(0) = 0$, then $f \equiv 0$*
2. *if $f''(0) < 0$, then f and f' are strictly decreasing functions*

Proof. We first prove case (1). At a critical point $s^* > 0$ such that $f'(s^*) = 0$ we have by (1.3.3) that $\Delta f(s^*) - |\nabla f(s^*)|^2 = f''(s^*) = 2f''(0) < 0$. This in conjunction with the boundary condition $f'(0) = 0$ proves the monotonicity of f. Noting that in local coordinates the evolution equation (1.3.3) for f reads

$$f'' + \left(\frac{a'}{a} + 2n\frac{b'}{b}\right) f' - (f')^2 = 2f''(0) \tag{1.4.1}$$

by the expression (2.2.4) for the Laplacian and

$$\left(\frac{a'}{a} + 2n\frac{b'}{b}\right)' = f'' - \left(\left(\frac{a'}{a}\right)^2 + 2n\left(\frac{b'}{b}\right)^2\right) \tag{1.4.2}$$

by the soliton equations (1.2.1) - (1.2.3), we obtain by differentiating (1.4.1) that

$$f''' = f'f'' + \left(\left(\frac{a'}{a}\right)^2 + 2n\left(\frac{b'}{b}\right)^2\right) f' - \left(\frac{a'}{a} + 2n\frac{b'}{b}\right) f'' \tag{1.4.3}$$

So whenever $f'' = 0$, we have

$$f''' = \left(\left(\frac{a'}{a}\right)^2 + 2n\left(\frac{b'}{b}\right)^2\right) f' < 0 \tag{1.4.4}$$

This proves that $f'' < 0$.

To prove case (2) note that the continuous dependence of the solution (f, a, b) on $f''(0)$ and case (1) imply that $f \leq 0$ everywhere. Since $a'(0) > 0$ we deduce that $\frac{a'}{a} + 2n\frac{b'}{b}$ is a Lipshitz function on any closed interval $I \subset (0, s_0)$ and hence, by standard theory of ODE, it follows from (1.4.1) that if f is constantly zero in a neighborhood of $s = 0$ it must be constantly zero on all of $[0, s_0)$. So we are left to show that f is zero near $s = 0$. If this were not the case, there would be an interval of the form $(0, \epsilon)$, $\epsilon > 0$ on which $f, f' < 0$. Furthermore, the boundary conditions (1.2.4) imply that $\frac{a'}{a} + 2n\frac{b'}{b} > 0$ on $(0, \epsilon)$, for ϵ sufficiently small. However, from (1.4.1) it would then follow that $f'' > 0$ on $(0, \epsilon)$ leading to a contradiction. □

Finally we can also prove that R is monotonically decreasing.

Lemma 1.4.3. *Let $s_0 \in \mathbb{R}_{>0}$ and $(f, a, b) : [0, s_0) \to \mathbb{R}^3$ be a solution to the soliton equations (1.2.1)-(1.2.3) with $f''(0) < 0$. Then R is a strictly decreasing function.*

Proof. From the evolution equation (1.3.5) for R we see that for any critical point $s^* > 0$ for which $R' = 0$ we have $\Delta R = R'' \leq -2|Ric|^2 = -2|\nabla_i \nabla_j f|^2 < 0$, where the last strict inequality follows from the previous lemma. Since $R' = 0$ and $R'' < 0$ at $s = 0$ we obtain the desired result. □

1.5 Existence of complete solitons

In this section we will prove the following theorem:

Theorem 1.5.1. *On the line bundle $L_{k,p}$ for $k, p \in \mathbb{N}$ there exists a family of complete steady Ricci solitons. In particular, there is a $f_0 \geq 0$ such that any $f''(0) < -f_0$ yields a solution $(f, a, b) : [0, \infty) \to \mathbb{R}^3$ to the soliton equations (1.2.1)-(1.2.3).*

Remark 1.5.2. In [Wink17] and [Stol17] these solitons were constructed independently.

The strategy will be to first show that as long as $Q < \sqrt{n+1}$ a solution cannot blow up in finite distance and then use the evolution equation (1.3.1) of Q to argue that we can make Q arbitrarily small by picking $f''(0) \ll -1$.

Lemma 1.5.3. *Let $s_0 > 0$ and $(f, a, b) : [0, s_0) \to \mathbb{R}^3$ be a solution to the soliton equations (1.2.1)-(1.2.3) with $f''(0) \leq 0$. If $Q < \sqrt{n+1}$ on $[0, s_0)$, the solution can be extended past s_0.*

Proof. By the monotonicity properties of a, b and f derived in lemma 1.4.1 we see that the condition $Q < \sqrt{n+1}$ implies

$$a'' \leq 2n\frac{a^3}{b^4} \leq \frac{2n(n+1)^{\frac{3}{2}}}{b} \leq 2n(n+1)^{\frac{3}{2}} \tag{1.5.1}$$

$$b'' \leq \frac{2n+2}{b} - 2\frac{a^2}{b^3} \leq 2\frac{n+1-Q^2}{b} \leq 2(n+1), \tag{1.5.2}$$

which in turn shows that

$$a'(s) < a'(0) + 2n(n+1)^{\frac{3}{2}}s \qquad a(s) < a'(0)s + n(n+1)^{\frac{3}{2}}s^2 \tag{1.5.3}$$

$$b'(s) < 2(n+1)s \qquad b(s) < 1 + (n+1)s^2 \tag{1.5.4}$$

as long as $Q < \sqrt{n+1}$ holds true. Furthermore, by lemma 1.4.2 and (1.4.1) it follows that

$$-\sqrt{-2f''(0)} \leq f' \leq 0. \tag{1.5.5}$$

Hence by the PicardLindelf theorem we can extend the solution past s_0.

□

Now we show that for at least short distance s we have $Q < \sqrt{n+1}$.

Lemma 1.5.4. *For any solution $(f, a, b) : [0, s_0) \to \mathbb{R}^3$ to the soliton equations (1.2.1)-(1.2.3) we have $Q(s) \leq a'(0)s$ for $s \leq \frac{1}{a'(0)}$. In particular we can extend the solution to $[0, \frac{1}{a'(0)}]$.*

Proof. From the evolution equation (1.3.1) for Q we have that whenever $0 \leq Q \leq 1$ and $Q' > 0$

$$[\ln Q']' \leq [f - (2n+1)\ln b]'. \tag{1.5.6}$$

Integrating we obtain

$$Q'(s) \leq Q'(0)\frac{e^{f(s)}}{b(s)^{2n+1}} \leq a'(0) \tag{1.5.7}$$

by the monotonicity properties of f and b, and the fact that $Q'(0) = a'(0)$. Integrating again, yields the desired result by lemma 1.5.3. □

Now we can prove theorem 1.5.1:

Proof of Theorem 1.5.1. From the soliton equations (1.2.1)-(1.2.3) it follows that

$$
\begin{aligned}
f'' &= \frac{a''}{a} + 2n\frac{b''}{b} && (1.5.8)\\
&= -2n\frac{a^2}{b^4} + \frac{4n(n+1)}{b^2} - 4n\frac{a'b'}{ab} - 2n(2n-1)\left(\frac{b'}{b}\right)^2 + \left(\frac{a'}{a} + 2n\frac{b'}{b}\right)f'. && (1.5.9)
\end{aligned}
$$

Solving the last equation for $\left(\frac{a'}{a} + 2n\frac{b'}{b}\right)f'$ and substituting the resulting expression into the evolution equation (1.4.1) of f shows that

$$
\begin{aligned}
f'' &= f''(0) - n\frac{a^2}{b^4} + \frac{2n(n+1)}{b^2} - 2n\frac{a'b'}{ab} - n(2n-1)\left(\frac{b'}{b}\right)^2 + \frac{(f')^2}{2}\\
&< f''(0) + 2n(n+1) + \frac{(f')^2}{2} && (1.5.10)
\end{aligned}
$$

for as long as a and b are increasing, which by lemma 1.5.4 is true for $s < \frac{1}{a'(0)}$. Now (1.5.10) implies that for constants $s_0 < \frac{1}{a'(0)}$ and $c_0 > 0$, we can find an $f_0 > 0$, such that for $f''(0) < -f_0$ we have $f'(s) \leq -c_0$ for $s > s_0$ and hence, by the monotonicity properties, $f(s) \leq -c_0(s - s_0)$ for $s > s_0$. Therefore from lemmas 1.5.4, the inequality (1.5.7) and the fact that $b \geq 1$ as long as $Q < \sqrt{n+1}$, we deduce that

$$
\begin{aligned}
Q &\leq a'(0)\left(s_0 + \int_{s_0}^{\infty} e^{-c_0(s-s_0)}\right) && (1.5.11)\\
&\leq a'(0)\left(s_0 + \frac{1}{c_0}\right).
\end{aligned}
$$

By choosing s_0 small and c_0 large we can ensure that $Q \leq 1$ for all times and thus lemma 1.5.3 yields the desired result. $\square$

Remark 1.5.5. From the above proof it follows that for any $c \in (0,1]$ there exists a $f_0 > 0$ such that for $f''(0) < -f_0$ we have $Q \leq c$.

1.6 Asymptotics

In this section we study the behavior of Q as $s \to \infty$ in the case that $f''(0) < 0$. The goal is to prove the following theorem:

Theorem 1.6.1. *Let $(f, a, b) : [0, \infty) \to \mathbb{R}^3$ be a solution to the soliton equations (1.2.1)-(1.2.3) with $f''(0) < 0$. Then either $\lim_{s\to\infty} Q = 0$ or $\lim_{s\to\infty} Q = 1$. Furthermore*

1. *if $\lim_{s\to\infty} Q = 0$ we have $a \sim const$ and $b \sim const\sqrt{s}$*

2. if $\lim_{s\to\infty} Q = 1$ *we have* $a \sim b \sim const\sqrt{s}$

as $s \to \infty$*. Finally, any complete solution to the soliton equations satisfies* $Q \leq 1$ *everywhere.*

For this it will be useful to rewrite the soliton equations (1.2.2) and (1.2.3) for a and b in the form

$$a'' = \frac{2nQ^4}{a} - 2n\frac{(a')^2}{a} + \left(f' + 2n\frac{Q'}{Q}\right)a' \tag{1.6.1}$$

$$b'' = 2\frac{n+1-Q^2}{b} - 2n\frac{(b')^2}{b} + \left(f' - \frac{Q'}{Q}\right)b'. \tag{1.6.2}$$

As the limits of both f' and Q as $s \to \infty$ exists, we will be able to derive the asymptotics of a and b from the following auxiliary lemma:

Lemma 1.6.2. *Let* $\epsilon > 0$ *and* $c_1^*, c_2^* > \epsilon$*. Assume* $c_i : [0,\infty) \to \mathbb{R}$*,* $i = 1,2$*, are two positive smooth functions satisfying*

$$|c_i(s) - c_i^*| < \epsilon,\ i = 1,2, \tag{1.6.3}$$

for all $s \geq 0$*. Then for a solution* $y : [0,\infty) \to \mathbb{R}$ *to the ODE*

$$y'' = \frac{c_1(s)}{2y} - 2n\frac{(y')^2}{y} - c_2(s)y' \tag{1.6.4}$$

with initial conditions $y(0), y'(0) > 0$ *there exists an* $s_0 > 0$ *such that for* $s > s_0$

$$y^2(s_0) + \gamma_-\left(1+\epsilon\right)^{-1}(s - s_0) \leq y^2(s) \leq y^2(s_0) + \gamma_+\left(s - s_0\right), \tag{1.6.5}$$

where

$$\gamma_\pm = \frac{c_1^* \pm \epsilon}{c_2^* \mp \epsilon}. \tag{1.6.6}$$

Proof. Note that by writing $z = y^{2n+1}$ the ODE (1.6.4) becomes

$$z'' = \frac{2n+1}{2}c_1(s)z^{\frac{2n-1}{2n+1}} - c_2(s)z' \tag{1.6.7}$$

Then defining $w = z'$ and $f(z,s) = \frac{2n+1}{2}\frac{c_1(s)}{c_2(s)}z^{\frac{2n-1}{2n+1}}$ we get the system of equations

$$z' = w \tag{1.6.8}$$

$$w' = c_2(s)\left(f(z,s) - w\right) \tag{1.6.9}$$

We will now investigate the phase diagram of this ODE system in the first quadrant $w > 0, z > 0$ (where we take z to be the x-axis and w to be the y-axis). Then consider the subregions in the first quadrant

$$\begin{aligned} R_- : &\quad 0 < w < f_-(z)(1+\epsilon)^{-1} \\ R_+ : &\quad w > f_+(z) \\ S : &\quad f_-(z)(1+\epsilon)^{-1} < w < f_+(z) \end{aligned}$$

where

$$f_{\pm}(z) = \frac{(2n+1)}{2}\frac{c_1^* \pm \epsilon}{c_2^* \mp \epsilon} z^{\frac{2n-1}{2n+1}} \equiv \frac{(2n+1)}{2}\gamma_{\pm} z^{\frac{2n-1}{2n+1}}$$

Note that we have $0 < f_-(z) < f(z,s) < f_+(z)$ if we pick $\epsilon > 0$ sufficiently small. In the Region R_- we have

$$\frac{\mathrm{d}w}{\mathrm{d}z} = c_2(s)\left(\frac{f(z,s)}{w} - 1\right) > (c_2^* - \epsilon)\,\epsilon \tag{1.6.10}$$

and in the subregion R_+

$$\frac{\mathrm{d}w}{\mathrm{d}z} = c_2(s)\left(\frac{f(z,s)}{w} - 1\right) < 0. \tag{1.6.11}$$

Because $f_+(z)$ is strictly increasing, any solution starting in R_+ will eventually enter S and never return to R_+. Similarly $\lim_{z\to\infty} f_-'(z) = 0$ implies that any solution starting in R_- will eventually leave R_-. We conclude that there exists an $s_0 > 0$ such that for $s > s_0$ $w(s), z(s)$ are in the region S. Thus for $s > s_0$

$$\frac{z'}{z^{\frac{2n-1}{2n+1}}} = \frac{2n+1}{2}\gamma(s)\left(1+\epsilon(s)\right)^{-1}, \tag{1.6.12}$$

where $\gamma(s)$ and $\epsilon(s)$ are functions in the range (γ_-, γ_+) and $(0, \epsilon)$ respectively. Integrating this equation from s_0 to s and re-substituting y we obtain the desired result. $\square$

We can prove a slight generalization to lemma 1.6.2, by considering the case in which $0 < c_1(s) \leq \epsilon$:

Corollary 1.6.3. *For the same assumptions in lemma 1.6.2, apart from $c_1^* = 0$ we have that there exists an $s_0 > 0$ such that for $s > s_0$*

$$y^2(s_0) \leq y^2(s) \leq y^2(s_0) + \frac{\epsilon}{c_2^* - \epsilon}(s - s_0). \tag{1.6.13}$$

Proof. First notice that y must be non-decreasing. This proves the lower bound. To prove the upper bound we can follow the proof of lemma 1.6.2, however this time we only consider the subregions in the first quadrant of the phase diagram

$$R_+ : \qquad w > f_+(z) \tag{1.6.14}$$

$$R_- : \qquad w \leq f_+(z) \tag{1.6.15}$$

for

$$f_+(z) = \frac{2n+1}{2}\frac{\epsilon}{c_2^* - \epsilon} z^{\frac{2n-1}{2n+1}}. \tag{1.6.16}$$

Any solution starting from R_+ will eventually enter R_- and remain there. Furthermore any solution starting from R_- remains in R_-. Therefore there exists a $s_0 > 0$, such that for $s > s_0$ we have

$$w \leq f_+(z) \tag{1.6.17}$$

Integrating this equation gives the desired result. $\square$

Now we can proceed to prove

Lemma 1.6.4. *For a solution $(f, a, b) : [0, \infty) \to \mathbb{R}^3$ to the soliton equations the limit $Q_\infty := \lim_{s\to\infty} Q$ exists. Furthermore, if $Q_\infty < \infty$ and $f''(0) < 0$ then $\lim_{s\to\infty} Q' = 0$.*

Proof. By lemma 1.3.1 we know that Q' changes its sign at most once and therefore the limit $Q_\infty = \lim_{s\to\infty} Q$ exists. To prove the second part note that the evolution equation (1.3.1) for Q can be written as

$$\left[b^{2n+1}e^{-f}Q'\right]' = (2n+2)b^{2n-1}e^{-f}\left(Q^3 - Q\right). \tag{1.6.18}$$

Integrating from 0 to s we obtain

$$Q'(s) = \frac{Q'(0)e^{f(s)}}{b(s)^{2n+1}} + (2n+2)\frac{e^{f(s)}}{b(s)^{2n+1}}\int_0^s b(t)^{2n-1}e^{-f(t)}\left(Q(t)^3 - Q(t)\right)\,\mathrm{d}t. \tag{1.6.19}$$

Since $f''(0) < 0$ by assumption, $\lim_{s\to\infty} f'(s) \equiv f'_\infty < 0$ by lemma 1.4.2. Moreover, the limit $b_\infty \equiv \lim_{s\to\infty} b$ exists by lemma 1.4.1.

In the case that $b_\infty = 0$, lemma 1.4.1 implies that there exists an $s_0 > 0$ such that for $s > s_0$ we have $Q^2 > n + 1$ and $b' < 0$. Therefore it follows that

$$\lim_{s\to\infty}\frac{e^f}{b^{2n-1}}\int_{s_0}^s b^{2n-1}e^{-f}\left(Q^3 - Q\right)\,\mathrm{d}s \geq 6\lim_{s\to\infty} e^f\int_{s_0}^s e^{-f}\,\mathrm{d}s = -\frac{6}{f'_\infty} > 0. \tag{1.6.20}$$

This however implies that the RHS of (1.6.19) tends to ∞ as $s \to \infty$, contradicting our assumption that $Q_\infty < \infty$.

In the case that $b_\infty = \infty$, it follows that b must be monotonically increasing. From this one can bound the RHS of (1.6.19) and show that $\lim_{s\to\infty} Q' = 0$.

Finally, in the case that $0 < b_\infty < \infty$, we can apply L'Hôpital's rule to the RHS of (1.6.19) and show that the limit

$$\lim_{s\to\infty} Q' = -(2n+2)\frac{1}{b_\infty^2 f'_\infty}\left(Q_\infty^3 - Q_\infty\right) \tag{1.6.21}$$

exists. Since $0 \leq Q_\infty < \infty$, we must have that $\lim_{s\to\infty} Q' = 0$. Furthermore we can deduce that $Q_\infty = 0$ or 1 in this case. □

For the rest of the chapter we will denote $Q_\infty = \lim_{s\to\infty} Q$ and $b_\infty = \lim_{s\to\infty} b$.

Lemma 1.6.5. *There are no complete solutions $(f, a, b) : [0, \infty) \to \mathbb{R}^3$ to the soliton equations (1.2.1)-(1.2.3) with $f''(0) < 0$ and $n + 1 < Q_\infty^2 < \infty$.*

Proof. Assume such a solution exists. Then by the proof of lemma 1.6.4 above, we know that $b_\infty = \infty$. Hence by lemma 1.4.1 b is monotonically increasing. Furthermore $Q' \geq 0$ and

$f' < 0$ by lemma 1.3.1 and 1.4.2. Let s_0 be such that for $s > s_0$ we have $Q^2 \geq n+1+c$, for some $c > 0$. Then from (1.6.2) it follows that

$$b'' < -\frac{2c}{b} \tag{1.6.22}$$

for $s > s_0$. Multiplying this inequality by b' and integrating from s_0 to s, we deduce

$$b'(s)^2 \leq b'(s_0)^2 - 4c \ln\left(\frac{b(s)}{b(s_0)}\right). \tag{1.6.23}$$

Therefore b' must become negative in finite distance, contradicting the monotonicity of b. □

Lemma 1.6.6. *For a complete solution $(f, a, b) : [0, \infty) \to \mathbb{R}^3$ to the soliton equations (1.2.1)-(1.2.3) with $Q_\infty < \infty$ and $f''(0) < 0$, we have either (i) $\lim_{s\to\infty} Q = 0$ or (ii) $\lim_{s\to\infty} Q = 1$.*

Proof. We will show that $Q_\infty \neq 0$ implies that $Q_\infty = 1$.

By lemma 1.6.5 we know that $Q_\infty^2 \leq n+1$. By lemma 1.6.4 we know that $\lim_{s\to\infty} Q' = 0$ and by the assumption $f''(0) < 0$ it follows that $\lim_{s\to\infty} f' = f'_\infty < 0$ by lemma 1.4.2.

First assume that $0 < Q_\infty^2 < n+1$. By applying lemma 1.6.2 to (1.6.1) and (1.6.2) we deduce that for any sufficiently small $\epsilon > 0$ there exists an $s_0 > 0$ such that for $s > s_0$

$$\frac{a(s_0)^2 + \gamma_{a,-}(1+\epsilon)^{-1}(s-s_0)}{b(s_0)^2 + \gamma_{b,+}(s-s_0)} \leq \frac{a^2(s)}{b^2(s)} \leq \frac{a(s_0)^2 + \gamma_{a,+}(s-s_0)}{b(s_0)^2 + \gamma_{b,-}(1+\epsilon)^{-1}(s-s_0)}, \tag{1.6.24}$$

where

$$\begin{aligned}\gamma_{a,\pm} &= \frac{4nQ_\infty^4 \pm \epsilon}{-f'_\infty \mp \epsilon}\\ \gamma_{b,\pm} &= \frac{4(n+1-Q_\infty^2) \pm \epsilon}{-f'_\infty \mp \epsilon}.\end{aligned}$$

Taking the limit of (1.6.24) as $s \to \infty$ we obtain

$$\frac{\gamma_{a,-}}{\gamma_{b,+}} \leq Q_\infty^2 \leq \frac{\gamma_{a,+}}{\gamma_{b,-}}. \tag{1.6.25}$$

Since $\epsilon > 0$ can be chosen arbitrarily small, we conclude that Q_∞ solves the equation

$$\frac{nQ_\infty^4}{n+1-Q_\infty^2} = Q_\infty^2, \tag{1.6.26}$$

which in the interval $(0, \sqrt{n+1})$ has the unique solution $Q_\infty = 1$.

In the case that $Q_\infty^2 = n+1$ it follows from corollary 1.6.3 that

$$Q^2(s) = \frac{a^2(s)}{b^2(s)} \geq \frac{a^2(s_0) + \gamma_{a,-}(1+\epsilon)^{-1}(s-s_0)}{b^2(s_0) + \gamma_{b,+}(s-s_0)} \tag{1.6.27}$$

However since $\gamma_{a,-} = O(1)$, $\gamma_{b,+} = O(\epsilon)$ and ϵ may be chosen arbitrarily small, this leads to a contradiction of $Q_\infty^2 = n + 1$.

$\square$

From the above proof one also sees that in both cases $Q_\infty = 0$ and $Q_\infty = 1$, $b \sim const\sqrt{s}$ as $s \to \infty$. It remains to study the asymptotics of a when $Q_\infty = 0$:

Lemma 1.6.7. *Let $(f, a, b) : [0, \infty) \to \mathbb{R}^3$ be a complete solution to the soliton equations (1.2.1)-(1.2.3) with $\lim_{s\to\infty} Q = 0$ and $f''(0) < 0$. Then $a_\infty := \lim_{s\to\infty} a < \infty$ and a is asymptotically constant.*

Proof. By lemma 1.6.2 in conjunction with (1.6.2) we have that $b \sim b_0\sqrt{s}$ as $s \to \infty$, where $b_0 = \sqrt{\frac{4(n+1)}{-f'_\infty}}$. Following the proof of lemma 1.6.2 one can also check that $b' \sim \frac{1}{2}b_0\frac{1}{\sqrt{s}}$ as $s \to \infty$ and therefore $\frac{b'}{b} \sim \frac{1}{2s} \to 0$ as $s \to \infty$. Furthermore $Q_\infty = 0$ implies that $Q'(s) < 0$ for s sufficiently large by lemma 1.3.1. Hence, from the evolution equation (1.3.1) for Q, it follows that for any $\epsilon > 0$ there exists an $s_0 > 0$ such that for $s > s_0$

$$Q'' + c_1 Q' \leq -\frac{c_2}{s}Q, \tag{1.6.28}$$

where $c_1 = -f'_\infty + \epsilon$ and $c_2 = -\frac{1}{2}f'_\infty - \epsilon$.

Claim 1: For any $\epsilon > 0$, we can find constants $C, s_0 > 0$ such that for $s > s_0$

$$Q(s) \leq Cs^{-\frac{1}{2}+\epsilon}. \tag{1.6.29}$$

Proof of Claim: Multiplying by the integrating factor $e^{c_1 s}$ and integrating we obtain

$$\begin{aligned} Q'(s) &\leq e^{c_1(s_0-s)}Q'(s_0) - c_2 e^{-c_1 s}\int_{s_0}^{s}\frac{Q(t)}{t}e^{c_1 t}\,\mathrm{d}t & (1.6.30)\\ &\leq e^{c_1(s_0-s)}Q'(s_0) - c_2\frac{Q(s)}{s}e^{-c_1 s}\int_{s_0}^{s}e^{c_1 t}\,\mathrm{d}t & (1.6.31)\\ &\leq e^{c_1(s_0-s)}\left(Q'(s_0) + \frac{c_2}{c_1}\frac{Q(s)}{s}\right) - \frac{c_2}{c_1}\frac{Q(s)}{s}. & (1.6.32)\end{aligned}$$

Since the first term decays exponentially we can choose s_0 large such that for $s > s_0$

$$Q'(s) \leq \left(-\frac{c_2}{c_1} + \epsilon\right)\frac{Q(s)}{s}. \tag{1.6.33}$$

Integrating this equation from s_0 to s shows that for $s > s_0$

$$Q(s) \leq Cs^{-\frac{c_2}{c_1}+\epsilon}, \tag{1.6.34}$$

where $C = Q(s_0)s_0^{\frac{c_2}{c_1}-\epsilon}$. $\blacksquare$

From the claim, the equation (1.6.1) and the monotonicity properties of a, b, f, it thus follows that for $\epsilon > 0$ there exist constants $C_1, C_2, s_0 > 0$ such that for $s > s_0$

$$a'' \leq \frac{2n}{a} Q^4 + a'f' \tag{1.6.35}$$
$$\leq \frac{C_1}{s^{2-\epsilon}} - C_2 a'. \tag{1.6.36}$$

Solving this differential inequality, one finds that a must be bounded as $s \to \infty$, which proves the desired result. □

Finally, we prove that for any global solution to the soliton equations (1.2.1)-(1.2.3) $Q \leq 1$ everywhere.

Lemma 1.6.8. *Let $(f, a, b) : [0, s_0] \to \mathbb{R}^3$ be a solution to the soliton equations (1.2.1)-(1.2.3) such that $Q(s_0) > 1$. Then the maximal extension of the solution (f, a, b) blows up in finite distance s.*

Proof. We will first show that b has to become monotonically decreasing. By lemma 1.6.6 we know that Q is unbounded and hence there exists an $s_1 > 0$ such that for $s > s_1$ we have $Q^2 > n + 2$. It follows from (1.2.3) that

$$b'' < -\frac{2}{b}, \tag{1.6.37}$$

for $s > s_1$. Multiplying this equation by b' and integrating we see that b' must become negative after finite distance s_2. Furthermore, by lemma 1.4.1, b' remains negative on the maximal extension of the solution. Thus for $s > s_2$ we have by equation (1.2.2) and the monotonicity properties of f that

$$a'' \geq c_1 a^3 - c_2 a', \tag{1.6.38}$$

where $c_1, c_2 > 0$. However, one can use phase diagrams to prove that any a with $a(s_2), a'(s_2) > 0$ satisfying this differential inequality must blow up in finite distance. We prove this using phase diagrams: Taking $z = a$, $w = a'$ we obtain the ODE system

$$z' = w \tag{1.6.39}$$
$$w' \geq c_2 \left(\frac{c_1}{c_2} z^3 - w \right) \tag{1.6.40}$$

Since $z' = a' > 0$ we can take z to be the independent variable and obtain

$$\frac{dw}{dz} \geq c_2 \left(\frac{c_1}{c_2} \frac{z^3}{w} - 1 \right). \tag{1.6.41}$$

Now take

$$g(z) = \frac{c_1}{c_2}\frac{z^3}{z^{\frac{3}{2}}+1} \tag{1.6.42}$$

and consider the regions

$$R_+ : w > g(z) \tag{1.6.43}$$
$$R_- : w < g(z) \tag{1.6.44}$$

in the first quadrant $w, z > 0$. If we are in R_- we have

$$\frac{dw}{dz} \geq c_2 z^{\frac{3}{2}} \tag{1.6.45}$$

and hence we cross over to region R_+ in finite z. Furthermore on the curve $w = g(z)$ we have $\frac{dw}{dz} > g'(z)$ for z large enough. Thus we have that $w(z)$ eventually remains in R_+. However, switching back to the independent variable s, this implies that eventually

$$z' \geq g(z), \tag{1.6.46}$$

which can easily be shown to blow up in finite time. □

Theorem 1.6.1 follows from above lemmas.

1.7 Existence of non-collapsed complete solitons

So far we have only shown the existence of gradient steady solitons with $Q_\infty = 0$ (see theorem 1.5.1 and remark 1.5.5). These solitons are collapsed and therefore cannot occur as blowup limits of Ricci flow on closed manifolds. In this section we will construct a complete steady soliton with $Q_\infty = 1$ in the case $k > p$. One can check that it is non-collapsed using theorem 1.6.1 of the previous section.

We begin by defining

$$f_0^* = \sup\{f_0 \in \mathbb{R} \mid \text{for } f''(0) \leq f_0, \text{a complete Ricci soliton exists}\}.$$

By theorem 1.6.1 we know that for any $f''(0) < f_0^*$ we must have $Q \leq 1$ everywhere. In the following we will show that $f_0^* < 0$ for $a'(0) = \frac{k}{p}(n+1) > n+1$ and then argue that choosing $f''(0) = f_0^*$ leads to a complete non-collapsed steady gradient Ricci soliton.

Lemma 1.7.1. *Let $(f, a, b) : 0 \in I \to \mathbb{R}^3$ be a maximal solution to the soliton equations (1.2.1)-(1.2.3) with initial conditions $a'(0) > n + 1$ and $f''(0) = 0$. Then $Q > 1$ in finite distance s.*

Proof. Note first that by lemma 1.4.2 we have $f \equiv 0$. By taking the change of variable $\frac{dr}{ds} = \frac{1}{p(r)}$ for $p : (0, \infty) \to \mathbb{R}$ some positive function, the soliton equations (1.2.1)-(1.2.3) become

$$0 = \frac{1}{a}\left(\frac{a'}{p}\right)' + 2n\frac{1}{b}\left(\frac{b'}{p}\right)' \tag{1.7.1}$$

$$\frac{1}{p}\left(\frac{a'}{p}\right)' = 2n\left(\frac{a^3}{b^4} - \frac{a'b'}{bp^2}\right) \tag{1.7.2}$$

$$\frac{1}{p}\left(\frac{b'}{p}\right)' = \frac{2n+2}{b} - 2\frac{a^2}{b^3} - \frac{a'b'}{ap^2} - (2n-1)\frac{1}{b}\left(\frac{b'}{p}\right)^2 \tag{1.7.3}$$

where a, b, f, p are viewed as functions of r and $'$ denotes differentiation with respect to r. These equations can be solved explicitly by taking the gauge

$$ap = L, \tag{1.7.4}$$

for $L > 0$ a constant (see [PP87]). Eliminating the term

$$\frac{1}{p}\left(\frac{a'}{p}\right)' \tag{1.7.5}$$

in equation (1.7.2) by the expression obtained for it from (1.7.1) we deduce that

$$b'' = -\frac{L^2}{b^3}. \tag{1.7.6}$$

One can check that this equation is solved by

$$b^2 = L^2 - r^2. \tag{1.7.7}$$

Substituting (2.3.3) into (1.7.3) and applying the gauge condition $pa = L$ we obtain the first order equation

$$\left[\frac{a^2(r^2 - L^2)^n}{(2n+2)L^2 r}\right]' = -\frac{(r^2 - L^2)^n}{r^2} \tag{1.7.8}$$

for a. Integrating, we therefore obtain the solution

$$a^2 = -(2n+2)L^2 r(r^2 - L^2)^{-n}\int_{r_b}^{r} \frac{(s^2 - L^2)^n}{s^2}\,\mathrm{d}s \tag{1.7.9}$$

$$b^2 = (L^2 - r^2) \tag{1.7.10}$$

$$p^2 = \frac{L^2}{a^2}, \tag{1.7.11}$$

where $-L < r_b < 0$ is some constant. From this we can compute

$$\frac{da}{ds}\Big|_{s=0} = \frac{1}{p(r)}\frac{da}{dr}\Big|_{r=r_b} = \frac{1}{2L}\frac{da^2}{dr}\Big|_{r=r_b} = -\frac{(n+1)L}{r_b} \tag{1.7.12}$$

$$b|_{s=0} = (L^2 - r_b^2) \tag{1.7.13}$$

showing that if we take

$$L = a'(0)\left(a'(0)^2 - (n+1)^2\right)^{-\frac{1}{2}} \tag{1.7.14}$$

$$r_b = -\frac{(n+1)L}{a'(0)} \tag{1.7.15}$$

the solution satisfies the initial conditions $\frac{da}{ds}\big|_{s=0} = a'(0)$ and $b\big|_{s=0} = 1$. However, taking the limit $r \to 0_-$ we see that

$$a^2 = (2n+2)L^2 > L^2 = b^2 \tag{1.7.16}$$

at $r = 0$. □

Now we can prove the existence of the non-collapsed steady Ricci soliton.

Theorem 1.7.2. *Let $(\hat{M}, J, \hat{g})$ be a Khler-Einstein manifold of positive scalar curvature. Then for $k > p(\hat{M}, \omega)$ there exists a non-collapsed steady gradient Ricci soliton on L_k with $\lim_{s\to\infty} Q = 1$.*

Proof. Choosing $f''(0) = f_0^*$ gives rise to a global solution $(a, b, f) : [0, \infty) \to \mathbb{R}^3$. If this were not the case, then Q would be greater than 1 after finite distance s by lemma 1.5.3. However, the set

$$\{f''(0) \in \mathbb{R} \mid a \text{ and } b \text{ cross in finite time.}\} \tag{1.7.17}$$

is open, because of the continuous dependence on the initial condition $f''(0)$. This contradicts the definition of f_0^*.

We will now show that for this solution $\lim_{s\to\infty} Q = 1$. Assume this were not true, then by lemma 1.6.6 $\lim_{s\to\infty} Q = 0$ and thus, by lemma 1.3.1, there exists a unique s_* such that $Q_{max} := \max_{s\in[0,\infty)} Q = Q(s_*) < 1$. We cannot have $Q_{max} = 1$, because this would imply that $a = b$ everywhere. Now take $s_{**} > s_*$ and note that by lemma 1.7.1 $f_0^* < 0$, since $a'(0) = (n+1)\frac{k}{p} > n+1$. Hence, by the continuous dependence of the solution on $f''(0)$, we can find an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ a solution $(f_\epsilon, a_\epsilon, b_\epsilon) : [0, s_{**}] \to \mathbb{R}^3$ with $f_\epsilon''(0) = f_0^* + \epsilon < 0$ exists and $Q_\epsilon := \frac{a_\epsilon}{b_\epsilon}$ obtains a local maximum $Q_{max,\epsilon} < 1$ at some $s_{*,\epsilon} < s_{**}$. From lemma 1.3.1 we deduce that $Q_\epsilon < 1$ on the maximal extension of the solution $(f_\epsilon, a_\epsilon, b_\epsilon)$. However by lemma 1.5.3 we deduce that $(f_\epsilon, a_\epsilon, b_\epsilon)$, $\epsilon < \epsilon_0$ can be extended to a complete solution, which contradicts the definition of f_0^*. □

This in conjunction with theorem 1.6.2 concludes the proof of our main theorem 1.1.2.

1.8 Taub-Nut like solitons and the Bryant soliton

As mentioned in the introduction, we can consider the completion of the warped product metric (1.1.9) on $\mathbb{R}_{>0} \times S^{2n+1}$ as a metric on $\mathbb{R}^{2n+2}$, by specifying the boundary conditions

$$a = b = 0 \qquad\qquad a' = b' = 1 \qquad \text{at } s = 0. \tag{1.8.1}$$

This is because near $s = 0$ the metric is then of the form

$$g \sim ds^2 + s^2 g_{S^{2n+1}}. \tag{1.8.2}$$

To ensure that the metric is smooth at $s = 0$ we need to further require that $a(s)$ and $b(s)$ are extendable to smooth odd functions around $s = 0$.

We would also like to point out that in the case of $a = b$ everywhere, we obtain a general rotationally symmetric metric on $\mathbb{R}^{2n+2}$ of the form

$$g = ds^2 + a(s)^2 g_{S^{2n+1}} \tag{1.8.3}$$

and the soliton equations (1.2.1) – (1.2.3) reduce to

$$f'' = (2n+1)\frac{a''}{a} \tag{1.8.4}$$

$$a'' = \frac{2n}{a}\left(1 - (a')^2\right) + a'f', \tag{1.8.5}$$

which are precisely the soliton equations of a rotationally symmetric gradient steady soliton on $\mathbb{R}^{2n+2}$. We will exploit this fact in theorem 1.8.1 below to give a another proof of the existence of the Bryant soliton in even dimensions greater than four. Furthermore, if we take $d = 2n + 2$ and allow $n \in \{\frac{k}{2} \mid k \in \mathbb{N}\}$ to take half-integer values in (1.8.4) and (1.8.5), we obtain the rotationally symmetric soliton equations on $\mathbb{R}^d$, $d \geq 3$. It will turn out that our proof of the existence of the Bryant soliton in theorem 1.8.1 carries over word by word to the odd dimensional case as well.

With boundary conditions (1.8.1), the soliton equations (1.2.1)-(1.2.3) are, as previously, degenerate at $s = 0$. Nevertheless one can adapt the proof of theorem 1.2.1 to show that for each $a_0, b_0 \in \mathbb{R}$ there exists a unique analytic solution near $s = 0$ satisfying $a'''(0) = a_0$ and $b'''(0) = b_0$ in addition to the above boundary conditions. Furthermore the solution depends smoothly on a_0 and b_0.

By applying L'Hôpital's rule to equation (1.2.1) we see

$$f''(0) = a'''(0) + 2nb'''(0) \tag{1.8.6}$$

and from (1.3.4) it follows that $R(0) = -2(n+1)f''(0)$. Since $R \geq 0$ for any ancient solution to Ricci flow we must therefore require that $a'''(0) + 2nb'''(0) \leq 0$.

All of our previous results carry over word by word or with slight modifications, allowing us to prove the following theorem with little extra effort:

Theorem 1.8.1. *Let $a_0 \le b_0$ such that $f_0 = a_0 + 2nb_0 \le 0$. Then there exists a complete solution $(f, a, b) : [0, \infty) \to \mathbb{R}^3$ to the soliton equations (1.2.1)-(1.2.3) with initial conditions $a = b = f = 0$, $a' = b' = 1$, $f' = 0$, $f'' = f_0$, $a''' = a_0$ and $b''' = b_0$ at $s = 0$. Furthermore there are three cases:*

1. *If $a_0 + 2nb_0 = 0$ and $a_0 = b_0$, we obtain the standard euclidean metric.*
2. *If $a_0 + 2nb_0 = 0$ and $a_0 < b_0$, we obtain a Taub-Nut metric with asymptotics $a \sim const$ and $b \sim s$.*
3. *If $a_0 + 2nb_0 < 0$ and $a_0 = b_0$, we obtain the Bryant soliton with asymptotics $a = b \sim const\sqrt{s}$.*
4. *If $a_0 + 2nb_0 < 0$ and $a_0 < b_0$, we obtain a Taub-Nut like Ricci soliton with asymptotics $a \sim const$ and $b \sim const\sqrt{s}$.*

Proof. In cases (1) and (2) we have $f''(0) = 0$ and hence $f \equiv 0$ everywhere by lemma 1.4.2. It easily seen that $a = b = s$ is the unique solution in case (1) and that it corresponds to the standard Euclidean metric on $\mathbb{R}^{2n+2}$. In case (2) we obtain the Taub-Nut metrics as derived in [AG03].

By L'Hôpital's rule we compute that

$$\lim_{s\to 0} Q = 1 \tag{1.8.7}$$

$$\lim_{s\to 0} Q' = 0 \tag{1.8.8}$$

$$\lim_{s\to 0} Q'' = a'''(0) - b'''(0) \tag{1.8.9}$$

Hence in case (4) we see that for small $s > 0$ we have $Q' < 0$. By lemma 1.3.1 it follows that $Q' < 0$ for as long as the solution exists and thus by lemma 1.5.3 we obtain a complete Ricci soliton $(f, a, b) : [0, \infty) \to \mathbb{R}^3$. From lemma 1.6.6 it follows that $\lim_{s\to\infty} Q = 0$ and therefore $a \sim const$ and $b \sim const\sqrt{s}$ as $s \to \infty$ by lemmas 1.6.7 & 1.6.4.

To prove case (3), assume that $Q > 1$ after finite distance s_0. Then, by the continuous dependence on parameters, we could pick an $\epsilon > 0$ sufficiently small such that the solution $(f_\epsilon, a_\epsilon, b_\epsilon) : [0, s_0] \to \mathbb{R}^3$ with initial conditions $a'''(0) = a_0 - \epsilon$ and $b'''(0) = b_0$ exists and $Q_\epsilon(s_0) := \frac{a_\epsilon}{b_\epsilon} > 1$. This however contradicts case (4). Therefore $Q \le 1$ everywhere and we obtain a complete solution $(f, a, b) : [0, \infty) \to \mathbb{R}^3$ by lemma 1.5.3. Now assume that $Q < 1$ after finite distance s_0. Then we could choose an $\epsilon > 0$ such that the solution $(f_\epsilon, a_\epsilon, b_\epsilon) : [0, s_0] \to \mathbb{R}^3$ with initial conditions $a'''(0) = a_0 + \epsilon$ and $b'''(0) = b_0$ exists and $Q_\epsilon(s_0) < 1$. However above calculation shows that $Q_\epsilon''(0) > 0$ and thus $Q_\epsilon'(s) > 0$ for $s > 0$ by lemma 1.3.1, leading to a contradiction. We conclude that $Q = 1$ and hence $a = b$ everywhere. Observe that we have thus found a solution to the equations (1.8.4) and (1.8.5). Hence this soliton must be the Bryant soliton. The asymptotics of the Bryant soliton follow easily from lemma 1.6.4. □

As mentioned above, the proof of the existence of the Bryant soliton in case (4) carries over word by word to odd dimensions $d \geq 3$ by allowing n to take half-integer values. One easily checks that all of the previous results also hold for such n. In particular, no term involving n in the evolution equation (1.3.1) of Q or any other (in-)equalities studied above changes its sign when we allow half-integer values for n. We only fail to have a geometrical interpretation of a and b, when $a \neq b$ in odd dimensions. When $a = b$, however, we can interprete a as in (1.8.3). Therefore we obtain another proof for the existence and uniqueness of the Bryant soliton in dimensions $d \geq 3$:

Theorem 1.8.2. *On $\mathbb{R}^d$, $d \geq 3$, there exists a unique rotationally symmetric gradient steady soliton, i.e. the Bryant soliton.*

1.9 Conjectures

In this section we briefly discuss two conjectures relating to the non-collapsed solitons from theorem 1.1.2. We numerically integrated the soliton equations (1.2.1) – (1.2.3) and found strong support for the following conjecture:

Conjecture 1. *For a line bundle L_k, $k > p(\hat{M}, \omega)$, the complete non-collapsed steady soliton of theorem 1.1.2 is unique up to scaling and isometry in the class of metrics (1.1.11).*

In particular, choosing the normalization $b(0) = 1$, there exists a $f_0^* \in \mathbb{R}$ such that

1. *if $f''(0) > f_0^*$, we obtain an incomplete metric*
2. *if $f''(0) = f_0^*$, we obtain a complete non-collapsed steady soliton*
3. *if $f''(0) < f_0^*$, we obtain a complete collapsed steady soliton*

Motivated by the discovery of the non-collapsed steady soliton in this chapter, we also conducted preliminary simulations of the full Ricci flow equation (1.1.1) to investigate whether these solitons appear as singularity models. Our results seem to indicate that they do indeed occur as blow-up limits. A paper [AW19] on this is in preparation. We therefore conjecture:

Conjecture 2. *The non-collapsed steady solitons of theorem 1.1.2 all occur as singularity models in Ricci flow.*

Note, also that in the case of L_1 on $\mathbb{C}P^1$ (i.e. $n = k = 1$ and $p = 2$ in our notation), Davi Maximo already showed in [M14] that the FIK shrinker, which is the unique shrinking Khler-Ricci soliton for a metric of the form (1.1.11), occurs as a singularity model.

In figures 1 and 2 you can see examples of complete solitons with $Q_\infty = 0$ and $Q_\infty = 1$ respectively.

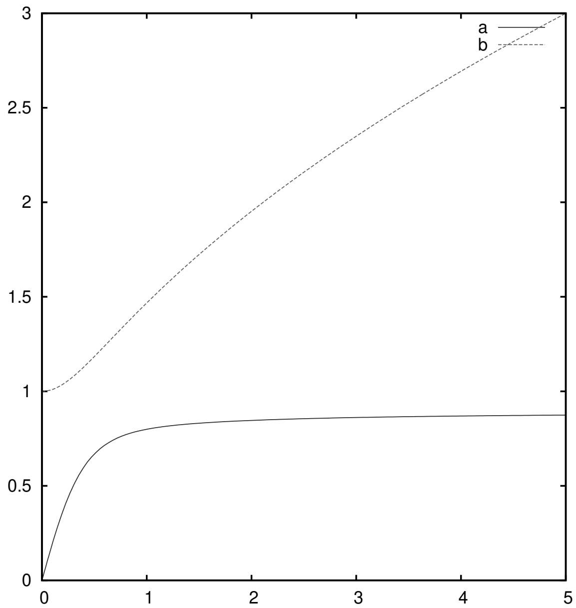

Figure 1.1: A collapsed soliton on $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_2$ with $f''(0) = -10$

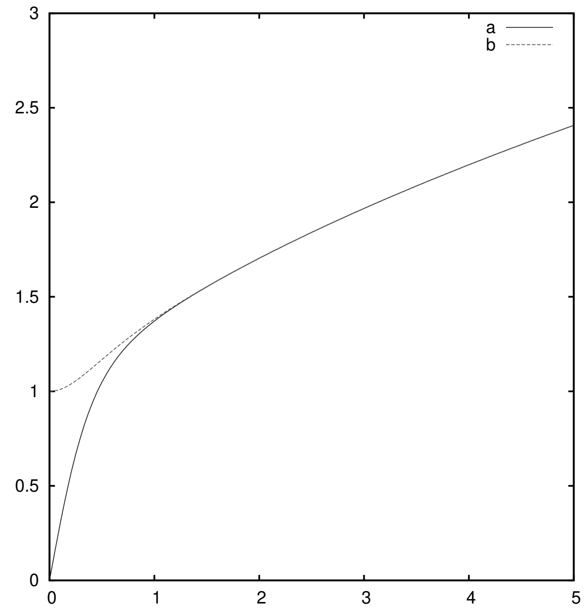

Figure 1.2: The non-collapsed soliton on $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_3$

1.10 Appendix A: Derivation of curvature tensor components

Here we derive the Ricci soliton equations. We will follow [PP87] to compute the Ricci tensor of the metric

$$g = ds^2 + a(s)^2 \left(d\tau - 2A\right)^2 + b(s)^2 \hat{g}, \tag{1.10.1}$$

on a complex line bundle of a Khler-Einstein manifold $(\hat{M}^{2n}, J, \hat{g})$, where A is a connection 1-form on $\hat{M}$ such that $dA = \omega$ and ω is Khler form of $\hat{M}$. We will assume that the metric $\hat{g}$ is scaled such that $Ric(\hat{g}) = 2(n+1)\hat{g}$, in order for a and b to have a nice geometrical interpretation when we choose $\mathbb{C}P^n$ equipped with the Fubini-Study metric as the base manifold. For the same reason we multiply the connection form A by 2.

We will compute the full curvature tensor of g using Cartan's formalism. Pick an orthonormal frame of 1-forms $e^0 = ds$, $e^1 = a\left(d\tau - 2A\right)$ and $e^i = b\hat{e}^i$, $i = 2, 3, \cdots, 2n+1$, where $\hat{e}^i$ is an orthonormal frame on the base $\hat{M}$. Denote by e_i, $i = 0, 1, \cdots, 2n+1$ and $\hat{e}_i$, $i = 2, 3, \cdots 2n+1$ the corresponding dual basis. In the following indices will run from either 0 to $2n+1$ or 2 to $2n+1$, which will be clear from context.

The connection 1-forms θ^i_j, defined by $\nabla e_i = \theta^j_i e_j$, and the curvature 2-forms Ω^j_i, defined by $R(\cdot,\cdot)e_i = \Omega_i^j e_j$, satisfy Cartan's structure equations

$$\begin{aligned} de^i &= -\theta^i_j \wedge e^j \\ \theta^i_j &= -\theta^j_i \\ \Omega^i_j &= d\theta^i_j + \theta^i_k \wedge \theta^k_j. \end{aligned}$$

Note that in coordinates we have $\Omega^i_j = \frac{1}{2}R^i{}_{jkm}e^k \wedge e^m$. In the following we will denote by $\hat{\theta}^i_j$ and $\hat{\Omega}^i_j$ the connection 1-forms and curvature 2-forms respectively, corresponding to the frame $\hat{e}^i$ on the base $(\hat{M}, \hat{g})$. Moreover $\hat{\nabla}$ will be the covariant derivative on $(\hat{M}, \hat{g})$. Hence we compute

$$\begin{aligned} \theta^0_1 &= -\frac{a'}{a}e^1 & \theta^0_i &= -\frac{b'}{b}e^i \\ \theta^1_i &= -\frac{a}{b^2}\omega_{ij}e^j & \theta^i_j &= \hat{\theta}^i_j + \frac{a}{b^2}\omega_{ij}e^1. \end{aligned}$$

Proceeding, we obtain

$$\begin{aligned} \Omega^0_1 &= -\frac{a''}{a}e^0 \wedge e^1 + \frac{1}{b^2}\left(a' - \frac{ab'}{b}\right) w_{ij}e^i \wedge e^j \\ \Omega^0_i &= -\frac{b''}{b}e^0 \wedge e^i + \frac{1}{b^2}\left(a' - \frac{ab'}{b}\right) w_{ij}e^1 \wedge e^j \\ \Omega^1_i &= \left(\frac{a^2}{b^4}\omega_{kj}\omega_{ki} - \delta_{ij}\frac{a'b'}{ab}\right) e^1 \wedge e^j - \frac{1}{b^2}\left(a' - \frac{ab'}{b}\right)\omega_{ij}e^0 \wedge e^j \\ \Omega^i_j &= \hat{\Omega}^i_j - \left(\frac{b'}{b}\right)^2 e^i \wedge e^j - \frac{a^2}{b^4}\left(\omega_{ij}\omega_{km} + \omega_{ik}\omega_{jm}\right) e^k \wedge e^m + \frac{2}{b^2}\left(a' - \frac{ab'}{b}\right)\omega_{ij}e^0 \wedge e^1 \end{aligned}$$

Note that we used that the complex structure J is parallel for a Khler manifold and thus $\omega_{ik}\hat{\theta}^k_j = \hat{\theta}^k_i\omega_{kj}$. Finally we can compute the non-zero entries of the Ricci tensor via $R_{ij} = R^k_{ikj}$

$$R_{00} = -\frac{a''}{a} - 2n\frac{b''}{b} \tag{1.10.2}$$

$$R_{11} = -\frac{a''}{a} + 2n\left(\frac{a^2}{b^4} - \frac{a'b'}{ab}\right) \tag{1.10.3}$$

$$R_{ii} = -\frac{b''}{b} + \frac{2n+2}{b^2} - 2\frac{a^2}{b^4} - \frac{a'b'}{ab} - (2n-1)\left(\frac{b'}{b}\right)^2 \tag{1.10.4}$$

From this we can also compute the scalar curvature

$$R = -2\frac{a''}{a} - 4n\frac{b''}{b} - 4n\frac{a'b'}{ab} - 2n(2n-1)\left(\frac{b'}{b}\right)^2 - 2n\frac{a^2}{b^4} + \frac{2n(2n+2)}{b^2} \tag{1.10.5}$$

Finally we need to compute the Hessian $\nabla^2 f$. From Koszul's formula it follows that the only non-zero terms are

$$\nabla^2_{e_0,e_0} = f'' \tag{1.10.6}$$

$$\nabla^2_{e_1,e_1} = \frac{a'}{a} f' \tag{1.10.7}$$

$$\nabla^2_{e_i,e_i} = \frac{b'}{b} f' \tag{1.10.8}$$

Therefore we obtain the soliton equations (1.2.1)-(1.2.3). From above it also follows that the Laplacian $\Delta\Phi$ of a function $\Phi : M \to \mathbb{R}$ depending only on s can be written as

$$\Delta\Phi = \Phi'' + \left(\frac{a'}{a} + 2n\frac{b'}{b}\right)\Phi'. \tag{1.10.9}$$

1.11 Appendix B: Existence of solutions to the Ricci soliton equation around $s = 0$

Here we prove theorem 1.2.1 ascertaining the local existence of analytic solutions to the soliton equations (1.2.1)-(1.2.3) around the origin. We begin by proving theorem 1.11.2, which generalizes the following result of the French mathematicians Briot and Bouquet to a parameter dependent system of ODEs:

Theorem 1.11.1 (Briot and Bouquet 1856, [BB1856])**.** *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an analytic function vanishing at $(0,0)$ and its derivative $\frac{\partial f}{\partial u}(0,0)$ not be a positive integer. Then there exists an analytical solution u around $r = 0$ to the non-linear ODE*

$$r\frac{du}{dr} = f(u,r). \tag{1.11.1}$$

We then show how the soliton equations (1.2.1)-(1.2.3) can be put in a form such that theorem 1.11.2 can be applied. This yields the proof of theorem 1.2.1.

Theorem 1.11.2. *Let $n \in \mathbb{N}$, $c \in \mathbb{R}$ and $U \subset \mathbb{R}^n$ an open subset containing the origin. Let*

$$\begin{aligned} P: \quad U \times \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R}^n \\ (u,r,\lambda) &\longrightarrow P(u,r,\lambda) \end{aligned} \tag{1.11.2}$$

be a vector valued analytic function around $(\vec{0},0,c)$ such that $P(\vec{0},0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$. If there is an open interval $I \ni c$ such that for all $\lambda \in I$ the matrix $\frac{\partial P}{\partial u}(\vec{0},0,\lambda)$ has no positive integer eigenvalues and

$$B = \sup_{\substack{\lambda\in I\\ m\in\mathbb{N}}} \left\| \left(mI_n - \frac{\partial P}{\partial u}(\vec{0},0,\lambda)\right)^{-1} \right\| < \infty, \tag{1.11.3}$$

then there exists an $\epsilon > 0$ *and a one-parameter family of analytic vector valued functions* $u(\cdot,\lambda) : (-\epsilon,\epsilon) \to \mathbb{R}^n$ *solving the ODE system*

$$r\frac{du(r,\lambda)}{dr} = P(u(r,\lambda), r, \lambda) \tag{1.11.4}$$

$$u(0,\lambda) = 0 \tag{1.11.5}$$

for $\lambda \in (c-\epsilon, c+\epsilon)$*. Furthermore* u *depends analytically on* λ*.*

Remark 1.11.3. 1. I_n denotes the $n \times n$ identity matrix.

2. For a matrix M we denote by $\|M\|$ the operator norm with respect to the standard Euclidean norm on $\mathbb{R}^n$.

Proof. We follow the proof of the one-dimensional case presented in [H79][Theorem 11.1]. Denote by u_i and P_i, $i = 1, 2, \cdots, n$, the components of u and P respectively. By analyticity we can write P_i as a power series around the origin

$$P_i(u, r, \lambda) = \sum_{k_1,\cdots,k_{n+2}\in\mathbb{N}} P_{ik_1\cdots k_{n+2}} u_1^{k_1}\cdots u_n^{k_n} r^{k_{n+1}} (\lambda - c)^{k_{n+2}} \tag{1.11.6}$$

such that for some $M > 0$, $R > 0$

$$|P_{ik_1\cdots k_{n+2}}| < \frac{M}{R^{k_1+\cdots+k_{n+2}}} \tag{1.11.7}$$

for $i = 1, 2, \cdots, n$ and $k_1, \cdots, k_{n+2} \in \mathbb{N}_0$. That is to say the powerseries converges whenever $|u_1|, \cdots, |u_n|, |r|, |\lambda - c| < R$. Defining the analytic functions

$$c_{ik_1\cdots k_{n+1}}(\lambda) := \sum_{k_{n+2}\in\mathbb{N}_0} P_{ik_1\cdots k_{n+2}}(\lambda - c)^{k_{n+2}} \tag{1.11.8}$$

we have that for $|\lambda - c| < \frac{R}{2}$

$$\left|c_{ik_1\cdots k_{n+1}}(\lambda)\right| < \frac{2M}{R^{k_1+\cdots k_{n+1}}}. \tag{1.11.9}$$

Letting

$$c_i(\lambda) = \frac{\partial P_i}{\partial r}(\vec{0}, 0, \lambda) \tag{1.11.10}$$

$$c_{ij}(\lambda) = \frac{\partial P_i}{\partial u_j}(\vec{0}, 0, \lambda) \tag{1.11.11}$$

we can then write

$$P_i(\vec{u}, r, \lambda) = c_i(\lambda)r + \sum_{j=1}^{n} c_{ij}(\lambda)u_j + \sum_{k_1+\cdots+k_{n+1}\geq 2} c_{ik_1\cdots k_{n+1}}(\lambda)u_1^{k_1}\cdots u_n^{k_n} r^{k_{n+1}} \tag{1.11.12}$$

for $i = 1, 2, \cdots, n$, whenever $|u_i|, r < R$ and $|\lambda - c| < \frac{R}{2}$. Below we fix such a λ and omit stating the dependence of our quantities on it.

We proceed by constructing a formal power series solution of the form

$$u_i(r) = \sum_{j=1}^{\infty} a_{ij} r^j \tag{1.11.13}$$

for $i = 1, 2, \cdots, n$ and $a_{ij} \in \mathbb{R}$. By substituting (1.11.13) into (1.11.4) we obtain

$$\begin{aligned}\sum_{j=1}^{\infty} j a_{ij} r^j = c_i r + \sum_{j=1}^{n} c_{ij} \left(\sum_{q=1}^{\infty} a_{jq} r^q \right) \\ + \sum_{k_1+\cdots+k_{n+1}\geq 2} c_{ik_1\cdots k_{n+1}} \left(\sum_{j=1}^{\infty} a_{k_1 j} r^j \right)^{k_1} \cdots \left(\sum_{j=1}^{\infty} a_{k_n j} r^j \right)^{k_n} r^{k_{n+1}}\end{aligned} \tag{1.11.14}$$

for $i = 1, 2, \cdots n$. By expanding and collecting terms of equal order we deduce that

$$\sum_{k=1}^{n} \left(\delta_{ik} - c_{ik}\right) a_{k1} = c_i \tag{1.11.15}$$

for the first order terms and

$$\sum_{k=1}^{n} \left(j\delta_{ik} - c_{ik}\right) a_{kj} = M_j(c_{ik_1k_2\cdots k_{n+1}}; \{a_{pq} \mid q \leq j-1, 1 \leq p \leq n\}) \tag{1.11.16}$$

for the j-th order terms ($j > 1$), where M_j is a multinomial with *non-negative* coefficients depending on the variables indicated. In the following denote by $D(m)$, $m \in \mathbb{N}$ the matrix with components

$$m\delta_{ij} - c_{ij}. \tag{1.11.17}$$

Because the matrix c_{ij} has no positive integer eigenvalues, $D(m)$ is invertible for all $m \in \mathbb{N}$ and we can uniquely determine a_{ij} order by order. In the following we will show that the resulting power series (1.11.13) has a positive radius of convergence.

For this consider an analytic vector valued function

$$G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \tag{1.11.18}$$

given by

$$G_i(\vec{u}, r) = C_i r + \sum_{k_1+\cdots+k_{n+1}\geq 2} C_{ik_1\cdots k_{n+1}} u_1^{k_1} \cdots u_n^{k_n} r^{k_{n+1}} \tag{1.11.19}$$

that majorizes P for all non-first order terms in u_i

$$|c_i| \leq C_i \tag{1.11.20}$$

$$|c_{ik_1\cdots k_{n+1}}| \leq C_{ik_1\cdots k_{n+1}} \tag{1.11.21}$$

and for which the Jacobian vanishes

$$\frac{\partial G}{\partial u}(\vec{0},0)=\vec{0}. \tag{1.11.22}$$

We choose

$$G_i(\vec{u},r)=\frac{2M}{\left(1-\frac{r}{R}\right)\left(1-\frac{1}{R}\left(u_1+\cdots+u_n\right)\right)}-2M\left(1+\frac{1}{R}\left(u_1+\cdots+u_n\right)\right) \tag{1.11.23}$$

in which case

$$C_1=C_2=\cdots=C_n \tag{1.11.24}$$

$$C_{1k_1\cdots k_{n+1}}=C_{2k_1\cdots k_{n+1}}=\cdots=C_{nk_1\cdots k_{n+1}} \tag{1.11.25}$$

for $k_1,\cdots,k_{n+1}\in\mathbb{N}$. We proceed by finding an analytic function

$$\begin{aligned} Y:\ \mathbb{R}&\longrightarrow\mathbb{R}^n \\ Y(0)&=\vec{0} \end{aligned} \tag{1.11.26}$$

solving the implicit equation

$$Y_i(r)=B\sqrt{n}G_i(Y(r),r),\ i=1,2,\cdots,n. \tag{1.11.27}$$

and show that it majorizes the formal power series solution found for u above, thereby proving the desired result. For our choice of G the equation (1.11.27) is quadratic and solved by

$$Y_i(r)=\frac{1-\sqrt{1-8\sqrt{n}MB\left(2\sqrt{n}MB\left(\frac{n}{R}\right)^2+\frac{n}{R}\right)\left(\frac{r}{r-R}\right)}}{4\sqrt{n}MB\left(\frac{n}{R}\right)^2+2\frac{n}{R}} \tag{1.11.28}$$

for $i=1,2,\cdots,n$. Note that Y_i vanishes at the origin and is analytic around $r=0$ with radius of convergence

$$R_c=\frac{R}{1+8\sqrt{n}MB\left(2\sqrt{n}MB\left(\frac{n}{R}\right)^2+\frac{n}{R}\right)}>0. \tag{1.11.29}$$

Therefore we can write Y as a power series

$$Y_i(r)=\sum_{j=1}^{\infty}A_{ij}r^j \tag{1.11.30}$$

for $i=1,2,\cdots,n$ and $|r|<R_c$. Because the Y_i, $i=1,2,\cdots,n$, are all equal we have

$$A_{1j}=A_{2j}=\cdots=A_{nj} \tag{1.11.31}$$

for $j \in \mathbb{N}$. Note that we can compute the A_{ij} by solving the implicit equation (1.11.27) order by order. This leads to

$$A_{i1} = B\sqrt{n}C_i \tag{1.11.32}$$

and

$$A_{ij} = B\sqrt{n}M_j(C_{ik_1k_2\cdots k_{n+1}}; \{A_{pq} \mid q \leq j-1, 1 \leq p \leq n\}) \tag{1.11.33}$$

for $j > 1$ and $i = 1, 2, \cdots, n$, where M_j is the same multinomial as above. This allows us to show by induction on j that

$$|a_{ij}| \leq A_{ij} \tag{1.11.34}$$

for $i = 1, 2, \cdots, n$ and $j \in \mathbb{N}$. In particular, notice that

$$|a_{i1}| = |\sum_{j=1}^{n} D(1)^{-1}_{ij} c_j| < B\sqrt{n}C_1 = A_{i1}, \tag{1.11.35}$$

where we used the assumption that

$$\left\|D^{-1}(m)\right\| \leq B \tag{1.11.36}$$

for $m \in \mathbb{N}$. By induction

$$\begin{aligned} |a_{ij}| &= \left|\sum_{q=1}^{n} D(j)^{-1}_{iq} M_j(c_{qk_1k_2\cdots k_{n+1}}; \{a_{pq} \mid q \leq j-1, 1 \leq p \leq n\})\right| && (1.11.37)\\ &\leq B\sqrt{n}M_j(C_{1k_1k_2\cdots k_{n+1}}; \{A_{pq} \mid q \leq j-1, 1 \leq p \leq n\}) \\ &= A_{ij} && (1.11.38) \end{aligned}$$

Hence the formal power series solution for u converges with radius of convergence greater or equal to R_c. Because R_c does not depend on λ as long as $|\lambda - c| \leq \frac{R}{2}$ and the coefficients a_{ij} depend analytically on λ we deduce that u varies analytically with λ. □

Now we can prove theorem 1.2.1:

Proof of theorem 1.2.1. Since $a'(s) \neq 0$, locally at $s = 0$ we can take a as the independent variable of the soliton equations (1.2.1)-(1.2.3) by considering the following change of variables

$$g = \frac{da^2}{h(a^2)} + g_{a,b(a)}. \tag{1.11.39}$$

Therefore taking $r = a^2$, we have

$$\frac{dr}{ds} = 2\sqrt{rh(r)} \tag{1.11.40}$$

and if we write $\dot{b}$ for $\frac{\partial b}{\partial r}$ etc. our soliton equations read

$$\ddot{f} = \frac{1}{4r}\frac{\dot{h}}{h} + 2n\frac{\ddot{b}}{b} + \frac{n}{r}\frac{\dot{b}}{b} + n\frac{\dot{h}\dot{b}}{hb} - \frac{1}{2r}\dot{f} - \frac{1}{2}\frac{\dot{h}}{h}\dot{f} \tag{1.11.41}$$

$$\dot{h} = \frac{2nr}{b^4} - 4nh\frac{\dot{b}}{b} + 2h\dot{f} \tag{1.11.42}$$

$$\ddot{b} = \frac{n+1}{2rhb} - \frac{1}{2hb^3} - \frac{\dot{b}}{r} - \frac{1}{2}\frac{\dot{h}}{h}\dot{b} - (2n-1)\frac{\dot{b}^2}{b} + \dot{f}\dot{b} \tag{1.11.43}$$

with boundary conditions

$$b(0) = 1 \tag{1.11.44}$$

$$\dot{b}(0) = \frac{n+1}{2a_0^2} \tag{1.11.45}$$

$$h(0) = a_0^2 \tag{1.11.46}$$

$$f(0) = 0 \tag{1.11.47}$$

$$\dot{f}(0) = \frac{f''(0)}{2a_0^2} \equiv c \tag{1.11.48}$$

Note that for fixed $n \in \mathbb{N}$ and $a_0 \in \mathbb{R}_{>0}$ we can freely vary $\dot{f}(0) = c$. The boundary condition for $\dot{b}$ was derived by using the L'Hôpital's Rule and noting that (1.2.3) at $s = 0$ implies that $b''(0) = n+1$. Since only $\dot{f}$ and $\ddot{f}$ appear in the equation we may consider this ODE as first order in $\dot{f}$. Furthermore, defining $F = \dot{f}$ and $B = \dot{b}$ we can turn the equations (1.11.41)-(1.11.43) into a first order system of ODEs in (F, h, b, B)

$$r\dot{F} = -F^2 r + 4nr\frac{FB}{b} - 2n\frac{B}{b} - 2n(2n-1)r\frac{B^2}{b^2} \tag{1.11.49}$$
$$+ \frac{n(n+1)}{hb^2} - \frac{n}{2hb^4}\left(r + 2Fr^2\right)$$

$$r\dot{h} = \frac{2nr^2}{b^4} - 4nhr\frac{B}{b} + 2hrF \tag{1.11.50}$$

$$r\dot{b} = Br \tag{1.11.51}$$

$$r\dot{B} = \frac{n+1}{2hb} - \frac{r}{2hb^3} - B - \frac{Br^2 n}{hb^4} + \frac{rB^2}{b} \tag{1.11.52}$$

Defining $u(\cdot, c) = (u_1(\cdot, c), u_2(\cdot, c), u_3(\cdot, c), u_4(\cdot, c)) \equiv (F(\cdot) - c, h(\cdot) - h(0), b(\cdot) - b(0), B(\cdot) - B(0))$ for $c \in \mathbb{R}$ we obtain an ODE system with parameter c of the form

$$r\frac{du_i}{dr} = P_i(u, r, c) \tag{1.11.53}$$

$$u_i(0, c) = 0 \quad \text{for } i = 1, 2, 3, 4, \tag{1.11.54}$$

where P is an analytic function in the neighbourhood of the point $(\vec{0}, 0, c)$ in $\mathbb{C}^6$ and $P(\vec{0}, 0, c) = 0$. We compute $\frac{\partial P_i}{\partial u_j}$ at $(\vec{0}, 0, c)$ and obtain

$$\begin{bmatrix} 0 & -\frac{n(n+1)}{a_0^4} & -\frac{n(n+1)}{a_0^2} & -2n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{(n+1)}{2a_0^4} & -\frac{(n+1)}{2a_0^2} & -1 \end{bmatrix}. \tag{1.11.55}$$

This matrix has characteristic polynomial

$$\det(mI - \frac{\partial P}{\partial u}) = m^3(m+1), \tag{1.11.56}$$

which has no positive integer roots. Therefore the inverse

$$\left(mI - \frac{\partial P}{\partial u}\right)^{-1} = \begin{bmatrix} \frac{1}{m} & -\frac{n(n+1)}{a_0^4 m(m+1)} & -\frac{n(n+1)}{a_0^2 m(m+1)} & -\frac{2n}{m^2+m} \\ 0 & \frac{1}{m} & 0 & 0 \\ 0 & 0 & \frac{1}{m} & 0 \\ 0 & -\frac{n+1}{2a_0^4 m(m+1)} & -\frac{n+1}{2a_0^2 m(m+1)} & \frac{1}{m+1} \end{bmatrix} \tag{1.11.57}$$

exists for $m \in \mathbb{N}$. Furthermore we can find a $B \in \mathbb{R}$ such that

$$\left\| \left(mI - \frac{\partial P}{\partial u}\right)^{-1} \right\| < B \tag{1.11.58}$$

for all $m \in \mathbb{N}$. Therefore we can apply theorem 1.11.2 proving the desired result. $\square$

Chapter 2

$U(2)$-invariant 4d Ricci flow singularities

2.1 Introduction

The main result of this chapter is to show

> *A Ricci flow on a four dimensional non-compact manifold may develop a Type II singularity modeled on the Eguchi-Hanson space in finite time.*

The Eguchi-Hanson space is diffeomorphic to the cotangent bundle of the two-sphere and asymptotic to the flat cone $\mathbb{R}^4/\mathbb{Z}_2$. It is the simplest example of a Ricci flat asymptotically locally euclidean (ALE) manifold and in the physics literature known as a gravitational instanton. The Eguchi-Hanson singularities constructed here are the first examples of orbifold singularities in Ricci flow, and are also the first examples of singularities with Ricci flat blow-up limits. As a byproduct of our work we also show that

> *A Ricci flow on a four dimensional non-compact manifold may collapse an embedded two-dimensional sphere with self-intersection $k \in \mathbb{Z}$ to a point in finite time and thereby produce a singularity.*

The singularities we construct when $|k| \geq 3$ are of Type II and the author conjectures that their blow-up limits are homothetic to the steady Ricci solitons found in Chapter 1.

Background

A family of time-dependent metrics $g(t)$ on a manifold M is called a Ricci flow if it solves the equation

$$\partial_t g(t) = -2Ric_{g(t)}. \tag{2.1.1}$$

Here $Ric_{g(t)}$ is the Ricci tensor of the metric $g(t)$. In local coordinates the Ricci flow equation can be written as a coupled system of second order non-linear parabolic equations. Heuristically speaking, the Ricci flow smoothens the metric $g(t)$, while simultaneously shrinking positively curved and expanding negatively curved directions at each point of the manifold.

Ricci flow was introduced by Hamilton [Ham82] in 1982 to prove that a closed three dimensional manifold admitting a metric of positive Ricci curvature also admits a metric of constant positive sectional curvature. This success demonstrated the power of Ricci flow and ignited much research in this area, culminating in Perelman's proof of the Poincaré and Geometrization Conjectures for three dimensional manifolds.

Even though every complete Riemannian manifold (M, g) of bounded curvature admits a short-time Ricci flow starting from g, singularities may develop in finite time. Understanding their geometry is central to the study of Ricci flow and has topological implications. For instance, Perelman proved the Geometrization Conjecture by analyzing the singularity formation in three dimensional Ricci flow and showing that a Ricci flow nearing its singular time exhibits one of the following two behaviors:

- Extinction: The manifold becomes asymptotically round before shrinking to a point
- (Degenerate or non-degenerate) Neckpinch: A region of the shape of a small cylinder $\mathbb{R} \times S^2$ develops

Based on this knowledge Perelman was able to construct a Ricci flow with surgery, which performs the decomposition of a three manifold into pieces corresponding to the eight Thurston geometries, yielding a proof of the Geometrization Conjecture.

In order to understand the formation of singularities in Ricci flow it is very useful to take blow-up limits. Roughly speaking one zooms into the region in which the singularity is forming by parabolically rescaling space and time. The resulting blow-up limit is an ancient Ricci flow called the **singularity model**. It encapsulates most of the geometric information of the singularity. Note that a Ricci flow is called **ancient** if it can be extended infinitely into the past. To date all known singularity models are either shrinking or steady **Ricci solitons**. These are self-similar solutions to the Ricci flow equation that, up to diffeomorphism, homothetically shrink or remain steady, and can be understood as a generalization of Einstein manifolds of positive or zero scalar curvature, respectively. Hamilton distinguishes between **Type I and Type II singularities**, depending on the rate at which the curvature blows up to infinity as one approaches the singular time. It has been proven that Type I singularities are modeled on shrinking Ricci solitons [EMT11], however it is unknown whether all Type II singularity models are steady Ricci solitons. In three dimensions the only Type I singularity models are S^3, $\mathbb{R} \times S^2$ and their quotients.

As three dimensional singularity formation is now well understood the next step is to analyze the four dimensional case, where currently very little is known other than that the possibilities are far more numerous. Below we list all Type I singularity models known in four dimensions:

1. $S^3 \times \mathbb{R}$ and its quotients

2. $S^2 \times \mathbb{R}^2$ and its quotients

3. Einstein manifolds of positive scalar curvature (e.g. S^4, $\mathbb{C}P^2$, etc.)

4. Compact gradient shrinking Ricci solitons that are not Einstein

5. The FIK shrinker [FIK03]

Note that (1) and (2) are just products of a three dimensional Type I singularity model with the real line. As Einstein manifolds in four dimensions remain to be classified, list item (3) may contain a very large set of manifolds. As for (4), to date only few examples of compact shrinking Ricci solitons are known and a list of these can be found in Cao's survey [Cao10]. The FIK shrinker is a non-compact $U(2)$-invariant shrinking Kähler-Ricci soliton, which is diffeomorphic to the blow-up of $\mathbb{C}^2$ at the origin. It is an open question whether there are other non-flat one-ended shrinking Ricci solitons in four dimensions. Maximo proved that Type I singularities modeled on the FIK shrinker may occur in $U(2)$-invariant Kähler-Ricci flow [M14].

The FIK shrinker models an interesting singularity in four dimensional Ricci flow — namely the collapse of an embedded two-dimensional sphere with non-trivial normal bundle. Topologically, real rank 2 vector bundles over the two-dimensional sphere are classified by their Euler class, which is an integer multiple of the generator of $H^2(S^2, \mathbb{Z})$. We call this multiple the twisting number and denote it by k. Recall that the self-intersection of an embedded two-dimensional sphere in a four-dimensional manifold is equal to the twisting number of its normal bundle. Unlike in Kähler geometry, where there is a canonical choice for the generator of $H^2(S^2, \mathbb{Z})$ and the sign of the self-intersection number is crucial, in the Riemannian case only its absolute value affects the geometry and behavior of embedded two-spheres under Ricci flow. Heuristically speaking, the larger $|k|$, the more negative curvature there is in the vicinity of the sphere and the less likely it collapses to a point. In the list above $S^2 \times \mathbb{R}^2$ and the FIK shrinker model the collapse of two dimensional spheres with self-intersection equal to 0 and -1, respectively. The main goal of Chapter 2 is to show that embedded spheres of self-intersection number $|k| \geq 2$ may also collapse in finite time. To explain this in greater detail we give an overview of our setup below.

Overview of setup

Let M_k, $k \geq 1$, be diffeomorphic to the blow-up of $\mathbb{C}^2/\mathbb{Z}_k$ at the origin, and denote by S^2_o the two-sphere stemming from the blow-up. Alternatively one can also view M_k as a plane bundle over S^2_o. Fix an arbitrary point o, for 'origin', on S^2_o. Note that S^2_o, with respect to the orientation inherited from $\mathbb{C}^2$, has self-intersection $-k$. Then equip M_k with an $U(2)$-invariant metric g. It turns out that with help of the Hopf fibration

$$\pi : S^3/\mathbb{Z}_k \to S^2$$

such $U(2)$-invariant metrics can be conveniently written as a warped product metric of the form

$$g = ds^2 + a^2(s)\omega \otimes \omega + b^2(s)\pi^* g_{S^2(\frac{1}{2})}, \tag{2.1.2}$$

on the open dense subset $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_k \subset M_k$. ω is the 1-form dual to the vertical directions of the Hopf fibration and s is a parametrisation of the $\mathbb{R}_{>0}$ factor. Note that g pulls back to a Berger sphere metric on the cross-sections $S^3/\mathbb{Z}_k$. One can complete g to a smooth metric on all of M_k by requiring

$$\begin{aligned} a(0) &= 0 \\ a_s(0) &= k \\ b(0) &> 0 \end{aligned} \tag{2.1.3}$$

and that $a(s)$ and $b(s)$ can be extended to an odd and even function around $s = 0$, respectively. Via the boundary condition $a_s(0) = k$ is how topology enters the analysis of the Ricci flow equation. We would like to mention here that throughout this chapter we often take the warping functions a and b to be functions of points (p, t) in spacetime rather than of (s, t). This will always be clear from context.

An upshot of writing the metric g in the form (2.1.2) is that the Ricci flow equation (2.1.1) reduces to a (1+1)-dimensional system of parabolic equations for the warping functions a and b, which simplifies the analysis greatly. In addition to this, both the FIK shrinker, which is diffeomorphic to M_1, and the Eguchi-Hanson space, which is diffeomorphic to M_2, are $U(2)$-invariant, and therefore their metrics can be written in the form (2.1.2). Here we only study Riemannian manifolds diffeomorphic to M_k, $k \in \mathbb{N}$, equipped with a $U(2)$-invariant metric of the form (2.1.2).

We will consider numerous *scale-invariant* quantities, the most fundamental and important of which we introduce here:

$$\begin{aligned} Q &:= \frac{a}{b} \\ x &:= a_s + Q^2 - 2 \\ y &:= b_s - Q \end{aligned}$$

The quantity Q measures the 'roundness' of the cross-sectional $S^3/\mathbb{Z}_k$. That is, when $Q = 1$ the metric on the cross-section is round. The quantities x and y are more interesting, as they measure the deviation of the metric g from being Kähler. In particular, when

$$y = 0$$

the manifold (M_k, g), $k \geq 1$, is Kähler with respect to the standard complex structure induced from $\mathbb{C}^2$. Moreover, when

$$x = y = 0$$

the manifold (M_k, g) is hyperkähler, and as we show in section 2.4, homothetic to the Eguchi-Hanson space.

Overview of results

The first main result of Chaper 2 is to show that

> *For a large class of $U(2)$-invariant asymptotically cylindrical initial metrics on M_k, $k \geq 2$, the Ricci flow develops a Type II singularity in finite time, as the area of S_o^2 decreases to zero. When $k = 2$ the blow-up limit of the singularity is homothetic to the Eguchi-Hanson space.*

We define the class of metrics for which this result holds in subsection 2.1. Note that in the $k = 2$ case the Eguchi-Hanson space is the first example of a Ricci flat singularity model. Based on numerical simulations the author believes that the Type II singularities in the $k \geq 3$ case are modeled on the steady solitons found in Chapter 1. A paper on the numerical results is in preparation [AW19].

The above result should be contrasted with the behavior of a Ricci flow starting from a Kähler metric. It is well known that the Kähler condition is preserved by Ricci flow, and that for such a flow the area of a complex submanifold evolves in a fixed manner. In particular, if (M, g) is a Kähler manifold with Kähler form ω, then under Ricci flow the Kähler class evolves by

$$[\omega(t)] = [\omega(0)] - 4\pi t c_1(M),$$

where $c_1(M)$ is the first Chern class of M. If we integrate the above equation over a complex curve Σ in M then one sees that

$$|\Sigma|_t = |\Sigma|_0 - 4\pi t \langle c_1(M), [\Sigma] \rangle,$$

where $|\Sigma|_t$ denotes the area of Σ at time t. In the case that $M \cong M_k$, $\Sigma = S_o^2$ and g is Kähler, it was shown in [FIK03, Proof of Lemma 1.2] that

$$\langle c_1(M_k), [S_o^2] \rangle = 2 - k$$

and hence

$$|S_o^2|_t = |S_o^2|_0 - 4\pi t \, (2 - k) \, . \tag{2.1.4}$$

This shows that for a Kähler-Ricci flow $(M_k, g(t))$, $k \in \mathbb{N}$ the two sphere S_o^2 can only collapse to a point when $k = 1$. In fact, when $k = 2$ the area of S_o^2 is stationary and for $k \geq 3$ increasing. Maximo in [M14] uses the Kähler condition and (2.1.4) to show that an embedded sphere of self-intersection -1 may collapse to a point in finite time under Ricci flow. Note that in our construction the metrics are not assumed to be Kähler and hence we cannot rely on (2.1.4).

The second main result of this chapter is the classification of all possible blow-up limits in the $k = 2$ case, including those at larger distance scales from the tip of M_2. In particular, we show that

For a large class of $U(2)$-invariant asymptotically cylindrical initial metrics on M_2 any sequence of blow-ups subsequentially converges to one of the following spaces:

(i) The Eguchi-Hanson space

(ii) The flat $\mathbb{R}^4/\mathbb{Z}_2$ orbifold

(iii) The 4d Bryant soliton quotiented by $\mathbb{Z}_2$

(iv) The shrinking cylinder $\mathbb{R} \times \mathbb{R}P^3$

The blow-up limits (ii) and (iii) show that the Eguchi-Hanson singularity results in the formation of an orbifold point, which to our knowledge the first concrete example of such in four dimensional Ricci flow.

We expect that many of our methods generalize to the analysis of Ricci flow on other cohomogeneity one manifolds. These are manifolds that admit an action by isometries of a compact Lie group G for which the quotient is one dimensional. The author believes that this work could contribute towards a complete picture of Ricci solitons and ancient Ricci flows on cohomogeneity one manifolds in four dimensions.

Precise statement of results

Before presenting the main theorems of Chapter 2, we list the definition of a class $\mathcal{I}$ of metrics needed to state our results.

Definition 7.2. *For $K > 0$ let $\mathcal{I}_K$ be the set of all complete* bounded curvature *metrics of the form (2.1.2) on M_k, $k \geq 1$, with* positive injectivity radius *that satisfy the following scale-invariant inequalities:*

$$\begin{aligned} Q &\leq 1 \\ a_s, b_s &\geq 0 \\ y &\leq 0 \\ \sup a_s &< K \\ \sup |bb_{ss}| &< K \end{aligned}$$

Denote by $\mathcal{I}$ the set of metrics g such that for sufficiently large $K > 0$ we have $g \in \mathcal{I}_K$.

For any $k \in \mathbb{N}$ the set $\mathcal{I}$ of metrics on M_k is non-empty, as for example the metric on M_k defined by

$$\begin{aligned} a(s) &= Q = \tanh(ks) \\ b(s) &= 1 \end{aligned}$$

is contained in $\mathcal{I}$. Moreover, as we prove in Lemma 2.7.9, the class $\mathcal{I}_K$ of metrics is preserved by the Ricci flow for sufficiently large $K > 0$. We will mainly consider Ricci flows $(M_k, g(t))$, $t \in [0, T)$, starting from an initial metric $g(0) \in \mathcal{I}$.

Now we list the precise statements of the main results of this chapter.

Theorem 9.1 (Type II singularities)**.** *Let $(M_k, g(t))$, $k \geq 2$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ (see Definition 2.7.2) with*

$$\sup_{p \in M_2} b(p, 0) < \infty.$$

Then $g(t)$ encounters a Type II curvature singularity in finite time $T_{sing} > 0$ and

$$\sup_{0 \leq t < T_{sing}} (T_{sing} - t)\, b^{-2}(o, t) = \infty.$$

Furthermore, there exists a sequence of times $t_i \to T_{sing}$ such that the following holds: Consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(o, t_i)} g\left(t_i + b^2(o, t_i) t\right), \quad t \in \big[-b(o, t_i)^{-2} t_i, b(o, t_i)^{-2} \left(T_{sing} - t_i\right)\big).$$

Then $(M_k, g_i(t), o)$ subsequentially converges, in the pointed Gromov-Cheeger sense, to an eternal Ricci flow $(M_k, g_\infty(t), o)$, $t \in (-\infty, \infty)$. When $k = 2$ the metric $g_\infty(t)$ is stationary and homothetic to the Eguchi-Hanson metric.

Remark 2.1.1. We would like to make the following remarks:

1. In Theorem 9.1, case $k = 2$, we only prove that there exists a blow-up sequence which converges to the Eguchi-Hanson space. In Theorem 12.1 below we extend this result and show that in fact any blow-up around the tip of M_2 is homothetic to the Eguchi-Hanson space.

2. The initial metric $g(0) \in \mathcal{I}$ with $\sup_{p \in M} b(p, 0) < \infty$ is asymptotic to $\mathbb{R} \times S^3/\mathbb{Z}_k$, where $S^3/\mathbb{Z}_k$ is equipped with a squashed Berger sphere metric. This is because metrics in $\mathcal{I}$ satisfy $a_s, b_s \geq 0$ and $Q \leq 1$.

The second main result of Chapter 2 is the classification of all possible blow-up limits in the $k = 2$ case:

Theorem 12.1 (Blow-up limits)**.** *Let $(M_2, g(t))$, $t \in [0, T_{sing})$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ (see Definition 2.7.2) with $\sup_{p \in M_2} b(p, 0) < \infty$. Let (p_i, t_i) be a sequence of points in spacetime with $b(p_i, t_i) \to 0$. Passing to a subsequence, we may assume that we are in one of the four cases listed below.*

(i) $\lim_{i \to \infty} \frac{b(p_i, t_i)}{b(o, t_i)} < \infty$

(ii) $\lim_{i\to\infty} \frac{b(p_i,t_i)}{b(o,t_i)} = \infty$ *and* $\lim_{i\to\infty} b_s(p_i,t_i) = 1$

(iii) $\lim_{i\to\infty} \frac{b(p_i,t_i)}{b(o,t_i)} = \infty$ *and* $\lim_{i\to\infty} b_s(p_i,t_i) \in (0,1)$

(iv) $\lim_{i\to\infty} \frac{b(p_i,t_i)}{b(o,t_i)} = \infty$ *and* $\lim_{i\to\infty} b_s(p_i,t_i) = 0$

Consider the dilated Ricci flows

$$g_i(t) = \frac{1}{b^2(p_i,t_i)} g\left(t_i + b^2(p_i,t_i)t\right), \quad t \in [-b(p_i,t_i)^{-2}t_i, 0].$$

Then $(M_2, g_i(t), p_i)$, $t \in [-b(p_i,t_i)^{-2}t_i, 0]$, *subsequentially converges, in the Cheeger-Gromov sense, to an ancient Ricci flow* $(M_\infty, g_\infty(t), p_\infty)$, $t \in (-\infty, 0]$. *Depending on the limiting property of the sequence* (p_i,t_i) *we have:*

(i) $M_\infty \cong M_2$ *and* $g_\infty(t)$ *is stationary and homothetic to the Eguchi-Hanson metric*

(ii) $M_\infty \cong \mathbb{R}^4 \setminus \{0\}/\mathbb{Z}_2$ *and* $g_\infty(t)$ *can be extended to a smooth orbifold Ricci flow on* $\mathbb{R}^4/\mathbb{Z}_2$ *that is stationary and isometric to the flat orbifold* $\mathbb{R}^4/\mathbb{Z}_2$

(iii) $M_\infty \cong \mathbb{R}^4 \setminus \{0\}/\mathbb{Z}_2$ *and* $g_\infty(t)$ *can be extended to a smooth orbifold Ricci flow on* $\mathbb{R}^4/\mathbb{Z}_2$ *that is homothetic to the 4d Bryant soliton quotiented by* $\mathbb{Z}_2$

(iv) $M_\infty \cong \mathbb{R} \times \mathbb{R}P^3$ *and* $g_\infty(t)$ *is homothetic to a shrinking cylinder*

Remark 2.1.2. Note that in Theorem 12.1 we do not prove that all blow-up limits (i)-(iv) occur. In fact, it may turn out that the Eguchi-Hanson singularity is isolated, in which case we would only see blow-up limits (i) and (ii).

As a byproduct of our work we also prove the following two theorems, which are of independent interest. Firstly, we exclude shrinking Ricci solitons in a large class of metrics.

Theorem 6.1 (No shrinker)**.** *On* M_k, $k \geq 2$, *there does not exists a complete* $U(2)$*-invariant shrinking Ricci soliton of bounded curvature satisfying the conditions*

1. $\sup_{p\in M_k} |b_s| < \infty$
2. $T_1 = a_s + 2Q^2 - 2 > 0$ *for* $s > 0$
3. $Q = \frac{a}{b} \leq 1$

As we show in section 5 the inequalities $T_1 > 0$ for $s > 0$ and $Q \leq 1$ are preserved by a Ricci flow $(M_k, g(t))$, $k \geq 2$, with $g(t) \in \mathcal{I}$. For this reason Theorem 6.1 can be used to exclude Type I singularities for such flows.

Secondly, we prove a uniqueness result for ancient Ricci flows on M_2.

Theorem 11.1 (Unique ancient flow). *Let $\kappa > 0$ and $(M_2, g(t))$, $t \in (-\infty, 0]$, be an ancient Ricci flow that is κ-non-collapsed at all scales and $g(t) \in \mathcal{I}$, $t \in (-\infty, 0]$ (see Definition 2.7.2). Then $g(t)$ is stationary and homothetic to the Eguchi-Hanson metric.*

We rely heavily on this result when we investigate all possible blow-up limits of a Ricci flow $(M_2, g(t))$ encountering a singularity at S^2_o.

Outline of Chapter 2 and proofs

Chapter 2 is organized by sections. Section 2 is preliminary and its goal is to set up in more detail the manifolds and metrics considered in Chapter 2. Here we also derive the full curvature tensor and Ricci flow equation for $U(2)$-invariant metrics. In section 3 we prove a maximum principle for degenerate parabolic differential equations on M_k. Beginning from section 4 we present new results. Below we outline the main results of those sections and their proofs.

Outline of section 4. A key ingredient in our work are the scale-invariant quantities

$$x = a_s + Q^2 - 2$$

and

$$y = b_s - Q,$$

that measure the deviation of a $U(2)$-invariant metric from being Kähler with respect to two fixed complex structures J_1 and J_2 on M_k, $k \geq 1$ (see section 2.4 for the precise definition of J_1 and J_2). In particular, a metric is Kähler with respect to J_1 whenever $y = 0$ and with respect to J_2 whenever $x = y = 0$.

Interestingly, a $U(2)$-invariant metric of the form (2.1.2) is Kähler with respect to J_2 if, and only if, the underlying manifold is diffeomorphic to M_2 and the metric is homothetic to the Eguchi-Hanson metric, as we show in Lemma 2.4.1. Therefore the quantities x and y can be used to measure how much a metric on M_2 deviates from the Eguchi-Hanson metric — a tool that is indispensable to our analysis. In the later sections we develop methods to control the behavior of x and y under the Ricci flow. This will allow us to prove that certain singularities of Ricci flows $(M_2, g(t))$ are modeled on the Eguchi-Hanson space.

In Lemma 2.4.2 of this section we also derive various properties of the Eguchi-Hanson metric. These are frequently used throughout the chapter.

Outline of section 5. The goal of this section is to derive various *scale-invariant* inequalities that are conserved by Ricci flow. We say that on a Riemannian manifold (M, g) a geometric quantity $T_g : M \to \mathbb{R}$ is scale-invariant if for every point $p \in M$, we have $T_g(p) = T_{\lambda g}(p)$ for all $\lambda > 0$. The scale-invariance of the inequalities derived is crucial, as it ensures that they pass to blow-up limits and thus also constrain their geometry.

We construct these inequalities from the scale-invariant quantities a_s, b_s and $Q := \frac{a}{b}$, where a and b are the warping functions of the metric g of the form (2.1.2). Note that subscript s denotes the derivative with respect to s. The key observation is that the evolution equation of the *scale-invariant* quantity

$$T_{(\alpha,\beta,\gamma)} = \alpha a_s + \beta Q b_s + \gamma Q^2, \quad \alpha, \beta, \gamma \in \mathbb{R},$$

can be written in the form

$$\partial_t T_{(\alpha,\beta,\gamma)} = \left[T_{(\alpha,\beta,\gamma)}\right]_{ss} + \left(2\frac{b_s}{b} - \frac{a_s}{a}\right)\left[T_{(\alpha,\beta,\gamma)}\right]_s + \frac{1}{b^2}C_{(\alpha,\beta,\gamma)},$$

where $C_{(\alpha,\beta,\gamma)}$ is a function of a_s, b_s and Q. For certain choices of α, β, γ and $\delta \in \mathbb{R}$ one can determine the sign of $C_{(\alpha,\beta,\gamma)}$ at a local extremum at which $T_{(\alpha,\beta,\gamma)} = \delta$. Depending on the sign, this allows one to prove, via the maximum principle, that either

$$T_{(\alpha,\beta,\gamma)} \geq \delta$$

or

$$T_{(\alpha,\beta,\gamma)} \leq \delta$$

is a conserved inequality. One of the conserved inequalities of this form is

$$x \leq 0,$$

however we derive many others.

In this section we also find conserved inequalities not of the above form. For instance, we show that each of the inequalities listed below are conserved by the Ricci flow:

- $Q \leq 1$
- $y \leq 0$
- $a_s, b_s \geq 0$

The proof is carried out by applying the maximum principle to their evolution equations or, in the case of $a_s, b_s \geq 0$, to their system of evolution equations. The conserved inequalities $Q \leq 1$, $y \leq 0$ and $a_s, b_s \geq 0$ are especially important, as they are part of the definition of the class of metrics $\mathcal{I}$ mentioned above, and constitute the first step in showing that $\mathcal{I}$ is preserved by the Ricci flow.

Outline of section 6. The main result of section 6 is Theorem 2.6.1, which rules out shrinking solitons on M_k, $k \geq 2$, within a large class of $U(2)$-invariant metrics. Before we outline the proof, note that from the evolution equation (2.2.12) of b under Ricci flow it follows by an application of L'Hôpital's rule that at $s = 0$

$$\partial_t b(0,t)^2 = 4\left(b y_s + k - 2\right). \tag{2.1.5}$$

This formula is a generalization of (2.1.4) to the non-Kähler case, as the area of S_o^2 at time t equals $b(o,t)^2\pi$. Hence a shrinking soliton must satisfy

$$\partial_t b(0,t)^2 < 0,$$

which for $k \geq 2$ implies that $y_s < 0$ at $s = 0$.

For the proof of Theorem 2.6.1 we have to rely on the inequality

$$T_1 = a_s + 2Q^2 - 2 \geq 0,$$

which by Lemma 2.5.8 is conserved by the Ricci flow. In particular, we show that amongst metrics of the form (2.1.2) on M_k, $k \geq 2$, satisfying $Q \leq 1$, $T_1 > 0$ when $s > 0$, and $\sup_{p \in M_k} |b_s| < \infty$ there are no shrinking solitons. We briefly sketch the proof here: First we show in Lemma 2.6.4 that $Q_s \geq 0$ for shrinking solitons. This follows from the Ricci soliton equation, which for metrics of the form (2.1.2) reduces to a system of ordinary differential equations. Then we consider the evolution equation

$$\partial_t y = y_{ss} + \frac{a_s}{a} y - \frac{y}{a^2} G, \tag{2.1.6}$$

of y, where G is a function of a_s, b_s and Q. In Lemma 2.6.5 we show that whenever $Q_s, T_1 > 0$ we have $G > 0$. This shows that under Ricci flow satisfying these inequalities a negative minimum of y is strictly increasing and a positive maximum is strictly decreasing. However, since y is a scale-invariant quantity, and a shrinking Ricci soliton, up to diffeomorphism, homothetically shrinks under Ricci flow, we see that the maximum or minimum of y must remain constant throughout the flow. We conclude that $y = 0$ everywhere, excluding a shrinking soliton. In the proof of Theorem 2.6.1, rather than working with the evolution equation (2.1.6) of y, we use the corresponding ordinary differential equation on a Ricci soliton background.

Outline of section 7. The goal of this section is to prove Theorem 2.7.5, which states that for a Ricci flow $(M_k, g(t))$, $k \geq 1$, starting from an initial metric $g(0) \in \mathcal{I}$ with $\sup_{p \in M_k} b(p,0) < \infty$ there exists a $C_1 > 0$ such that the curvature bound

$$|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{b^2}$$

holds. The proof is carried out by a contradiction/blow-up argument: Assume there exists a sequence of numbers $D_i \to \infty$ and points (p_i, t_i) in spacetime such that

$$K_i := |Rm_{g(t_i)}|_{g(t_i)}(p_i) = \frac{D_i}{b(p_i, t_i)^2}.$$

Consider the rescaled metrics

$$g_i(t) = K_i g\left(t_i + \frac{t}{K_i}\right), \quad t \in [-K_i t_i, 0],$$

normalized such that $|Rm_{g_i(0)}|_{g_i(0)}(p_i) = 1$. Then Perelman's no-local-collapsing theorem shows that $(M_k, g_i(t), p_i)$ subconverges to an ancient non-collapsed Ricci flow $(M_\infty, g_\infty(t), p_\infty)$, $t \leq 0$. As $D_i \to \infty$ the warping functions b_i corresponding to the metrics $g_i(t)$ satisfy $b_i(p_i, 0) \to \infty$. Recalling that the warping function b describes the size of the base manifold S^2 in the Hopf fibration of the $S^3/\mathbb{Z}_k$ cross-sections, one can see that $(M_\infty, g_\infty(t))$ splits as $M_\infty = \mathbb{R}^2 \times N$, where $\mathbb{R}^2$ is equipped with the flat euclidean metric and the restriction of $g_\infty(t)$ to N is a 2d non-compact κ-solution. However, the only κ-solutions in 2d are either the shrinking sphere or its $\mathbb{Z}_2$ quotient, both of which are compact. This is a contradiction and the proof of the curvature bound follows.

In Corollary 2.7.6 we show that ancient Ricci flows in $\mathcal{I}$, which are κ-non-collapsed at all scales, also satisfy the curvature bound

$$|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{b^2}.$$

This curvature bound will be important in section 2.11.

Outline of section 8. In this section we prove various local and global compactness results for $U(2)$-invariant Ricci flows in the class of metrics $\mathcal{I}$. To state the results we need to first introduce the following notation for a $U(2)$-invariant Riemannian manifold (M, g):

- Let $\Sigma_p \subset M$ denote the orbit of p under the $U(2)$-action.
- Let
$$C_g(p, r) := \left\{ q \in M \,\middle|\, d_g(q, \Sigma_p) < r \right\}$$

One sees that $C_g(p, r)$ is the tubular neighborhood of 'radial width' r of the orbit Σ_p of p under the $U(2)$-action. See Definition 2.2.1 for more details.

The main result of this section is Theorem 2.8.1, which states under which conditions a sequence of $U(2)$-invariant Ricci flows of the form $(C_{g_i(0)}(p_i, r), g_i(t), p_i)$, $t \in [-\Delta t, 0]$, $\Delta t > 0, r > 0$, subsequentially converges, in the Cheeger-Gromov sense, to a limiting $U(2)$-invariant Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$, $t \in [-\Delta, 0]$. Amongst other conditions, we require that $g_i(t)$ is

- κ-non-collapsed at some scale $\rho > 0$ at the point $(p_i, 0)$ in spacetime
- In the class $\mathcal{I}$
- Normalized such that $b(p_i, 0) = 1$
- Of uniformly bounded curvature in $C_{g_i(0)}(p_i, r) \times [-\Delta t, 0]$

We also show that after choosing suitable coordinates the warping functions of the metrics $g_i(t)$ subsequentially converge to the corresponding warping functions of $g_\infty(t)$. The compactness result of Theorem 2.8.1 is used frequently throughout the chapter, especially its variation, stated in Proposition 2.8.3.

Outline of section 9. The main goal of this section is to constrain the geometry of ancient Ricci flows $(M_k, g(t))$, $k \in \mathbb{N}$, $t \in (-\infty, 0]$ in the class of metrics $\mathcal{I}$ that are κ-non-collapsed at all scales. This is achieved by proving that various scale-invariant inequalities hold. For instance, in Theorem 2.9.1 we prove that three inequalities of the form $T_{(\alpha,\beta,\gamma)} \geq 0$, as in introduced in the outline of section 5 above, hold on such ancient flows. Furthermore, we prove in Theorem 2.9.2 that an ancient Ricci flow on M_2 in $\mathcal{I}$ which is Kähler with respect to J_1, i.e. $y = 0$ everywhere, is stationary and homothetic to the Eguchi-Hanson space. This result will be used in section 2.10, where we construct an eternal blow-up limit of a Ricci flow on M_2 that is homothetic to the Eguchi-Hanson space.

The proof of these theorems is via a **contradiction/compactness argument** frequently employed throughout the chapter. We briefly sketch the method here: Assume we want to prove that a scale-invariant inequality $T \geq 0$ holds on $M_k \times (-\infty, 0]$. We argue by contradiction and assume that

$$\iota := \inf_{M_k \times (-\infty, 0]} T < 0.$$

We then take a sequence of points (p_i, t_i) in spacetime such that $T(p_i, t_i) \to \iota$ as $i \to \infty$, and consider the dilated metrics

$$g_i(t) = \frac{1}{b(p_i, t_i)^2} g\left(t + t_i b(p_i, t_i)^2\right), \quad t \in [-\Delta t, 0],$$

on the tubular neighborhoods $C_{g_i(0)}(p_i, \frac{1}{2})$ (see Definition 2.2.1) for some small $\Delta t > 0$. By the compactness results of section 8, in particular Proposition 2.8.3, the Ricci flows $(C_{g_i(0)}(p_i, \frac{1}{2}), g_i(t), p_i)$, $[-\Delta t, 0]$, subsequentially converges to a Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$, $[-\Delta t, 0]$, where

$$T(p_\infty, 0) = \inf_{\mathcal{C}_\infty \times [-\Delta t, 0]} T = \iota < 0$$

by the scale invariance of T. If, however, the evolution equation of T precludes a negative infimum from being attained, we have arrived at a contradiction and proven the desired result.

Outline of section 10. The goal of this section is to prove Theorem 2.10.1, which states that a Ricci flow $(M_k, g(t))$, $k \geq 2$, starting from an initial metric $g(0) \in \mathcal{I}$ with $\sup_{p \in M_k} b(p, 0) < \infty$ encounters a Type II singularity in finite time at the tip of M_k as the area of S^2_o decreases to zero. In the $k = 2$ case we show that such a singularity possesses a blow-up limit that is stationary and homothetic to the Eguchi-Hanson space. We do not further investigate the $k \geq 3$ case, however the author conjectures that their blow-up limits are homothetic to the steady Ricci solitons found in Chapter 1.

The proof is carried out in multiple steps. First we show in Lemma 2.10.5 that $g(t)$ encounters a singularity in finite time $T_{sing} \in (0, \infty)$ and $b(o, t) \to 0$ as $t \to T_{sing}$. This shows that the two-sphere S^2_o at the tip of M_k collapses to a point in finite time and thereby produces a singularity.

In the second step, we rely on the results of section 6 to show that a blow-up limit around $o \in S_o^2$ cannot be a shrinking Ricci soliton when $k \geq 2$. As all Type I singularities are modeled on shrinking Ricci solitons we deduce that the singularity is of Type II.

In the third step we borrow a trick due to Hamilton to pick a sequence of times $t_i \to T_{sing}$ such that the following holds: Take the rescaled metrics

$$g_i(t) = \frac{1}{b(o,t_i)^2} g\left(t_i + b^2(o,t_i)t\right), \quad t \in \big[-b(o,t_i)^{-2}t_i, b(o,t_i)^{-2}\left(T_{sing} - t_i\right)\big),$$

where we recall that $o \in S_o^2$. Then $(M_k, g_i(t), o)$ subsequentially converges to an eternal Ricci flow $(M_\infty, g_\infty(t), o)$, $t \in (-\infty, \infty)$, where M_∞ is diffeomorphic to M_k.

In the final step we analyze the geometry of M_∞ when $k = 2$. It turns out that for the choice of times t_i it follows that

$$\partial_t b(o,0) = 0$$

on M_∞ background. By the evolution equation (2.1.5) of b at o this implies

$$y_s(o,0) = 0.$$

Applying a strong maximum principle we deduce that $y = 0$ everywhere. By the results of section 9 it then follows that $g_\infty(t)$ is stationary and homothetic to the Eguchi-Hanson metric.

We mention here that the $k = 2$ case of Theorem 2.10.1 is superseded by Corollary 2.11.2 of Theorem 2.11.1. However, since the proof of Theorem 2.10.1 is simpler we present it here.

Outline of section 11. The goal of this section is to show that an ancient Ricci flow $(M_2, g(t)), t \in (-\infty, 0]$, which is κ-non-collapsed at all scales and satisfies $g(t) \in \mathcal{I}$, is stationary and homothetic to the Eguchi-Hanson space. The most important consequence of this is that in Theorem 2.10.1 in fact *any* blow-up of the singularity forming at the tip of M_2 is homothetic to the Eguchi-Hanson space, whereas we had previously only proven that there exists *a* blow-up sequence that converges to the Eguchi-Hanson space.

The proof idea, which we call **successive constraining**, is to find a continuously varying family of preserved inequalities $Z_\theta \geq 0$, $\theta \in [0,1]$, for which $Z_0 \geq 0$ on $M_2 \times (-\infty, 0]$ implies that $g(t)$ is homothetic to the Eguchi-Hanson metric. For our choice of conserved inequalities $Z_\theta \geq 0$, $\theta \in [0,1]$, it follows from the work of section 9 that $Z_1 \geq 0$ on $M_2 \times (-\infty, 0]$. Then we deform the inequality $Z_1 \geq 0$ along the path $Z_\theta \geq 0$, $\theta \in [0,1]$, to the inequality $Z_0 \geq 0$ with help of the strong maximum principle applied to the evolution equation of Z_θ. This allows us to deduce that $g(t)$ is stationary and homothetic to the Eguchi-Hanson space. In subsection 2.11 we give a more detailed outline of the proof of Theorem 2.11.1.

Outline of section 12. The main result of this section is Theorem 2.12.1, which characterizes all the possible blow-up limits of a Ricci flow $(M_2, g(t))$ starting from an initial metric $g(0) \in \mathcal{I}$ with $\sup_{p \in M_2} b(p,0) < \infty$. We show that the only possible blow-up limits are (i) the Eguchi-Hanson space, (ii) the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$, (iii) the 4d Bryant soliton quotiented by $\mathbb{Z}_2$ and (iv) the shrinking cylinder $\mathbb{R} \times \mathbb{R}P^3$.

Below we give a brief outline of the proof of Theorem 2.12.1: Assume we are given a sequence of points (p_i, t_i) in spacetime with $b(p_i, t_i) \to 0$. Consider the rescaled metrics

$$g_i(t) = \frac{1}{b(p_i, t_i)^2} g(t_i + b(p_i, t_i)^2 t), \quad t \in [-b(p_i, t_i)^{-2} t_i, 0].$$

By passing to a subsequence we may assume that either

$$\text{(I)} \ \sup_i \frac{b(p_i, t_i)}{b(o, t_i)} < \infty \quad \text{or} \quad \text{(II)} \ \lim_{i \to \infty} \frac{b(p_i, t_i)}{b(o, t_i)} = \infty.$$

By section 11 we already know that in case (I) we converge to the Eguchi-Hanson space. Therefore we only need to investigate the behavior in case (II), i.e. at scales larger than the forming Eguchi-Hanson singularity. For this we need to divide case (II) into three subcases: By passing to a subsequence we may assume that

$$\text{(II.a)} \ b_s(p_i, t_i) \to 1 \ \text{ or } \ \text{(II.b)} \ b_s(p_i, t_i) \to \eta \in (0, 1) \ \text{ or } \ \text{(II.c)} \ b_s(p_i, t_i) \to 0.$$

For (II.a) and (II.c) we show in Lemma 2.12.9 and Lemma 2.12.6 that $(M_2, g_i(t), p_i)$ subsequentially converges to the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$ and the shrinking cylinder $\mathbb{R} \times \mathbb{R}P^3$, respectively. The proof of these lemmas is relatively easy. Proving in Lemma 2.12.8 that the blow-up limit in case (II.b) is homothetic to the 4d Bryant soliton quotiented by $\mathbb{Z}_2$ is trickier. Here we rely on Lemma 2.12.3, which characterizes the geometry of the high curvature regions of $g(t)$ at distance scales larger than the Eguchi-Hanson singularity away from the tip of M_2. In subsection 2.12 we give a more detailed outline of the proof of Theorem 2.12.1.

Further questions and conjectures

In this section we collect some conjectures and further questions that arise from our results. The central open question remaining in this chapter is whether or not the Eguchi-Hanson singularity of Theorem 2.12.1 is isolated. By isolated we mean that the only blow-up limits are the Eguchi-Hanson space and its asymptotic cone, the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$. We conjecture that

Conjecture 3. *The Eguchi-Hanson singularity of Theorem 2.12.1 is not isolated and all four blow-up limits (i)-(iv) occur. In particular, it is accompanied by a Type I singularity modeled on the shrinking cylinder* $\mathbb{R} \times S^3/\mathbb{Z}_2$.

An affirmative answer to this conjecture would provide evidence for a longstanding conjecture in Ricci flow stating that a Type II singularity is always accompanied by a Type I singularity in its vicinity. The author has an argument showing that if the Eguchi-Hanson singularity were isolated, the curvature would blow up at a rate faster than $(T_{sing} - t)^{-\lambda}$, where λ is any positive constant.

Although we have not analyzed the blow-up limits of a $U(2)$-invariant Ricci flow $(M_k, g(t))$, $t \in [0, T_{sing})$, in the $k \neq 2$ case, we believe that for each $k \in \mathbb{N}$ there exists a unique blow-up limit of the singularity arising from the collapse of the two sphere S^2_o at the tip of M_k. In collaboration with Jon Wilkening the author has already conducted numerical simulations confirming this, and a paper is in preparation [AW19]. In particular, we conjecture that

Conjecture 4. *Let $(M_k, g(t))$ be a $U(2)$-invariant Ricci flow encountering a singularity at the tip of M_k, as the area of S^2_o decreases to zero. Then the following picture holds:*

k	*Blow-up limit at $o \in S^2_o$*	*Type*	*Isolated*
1	*FIK shrinker*	*Type I*	*Yes*
2	*Eguchi-Hanson space*	*Type II*	*No*
≥ 3	*Steady Ricci solitons of Chapter 1*	*Type II*	*No*

By isolated we mean that the singularity is not accompanied by a Type I singularity in its vicinity. For instance, in the case $k \geq 2$ we expect a singularity caused by the collapse of the two-sphere S^2_o at the tip of M_k to always be accompanied by a Type I singularity modeled on the shrinking cylinder $\mathbb{R} \times S^3/\mathbb{Z}_k$ and therefore not to be isolated. If for each $k \geq 3$ the corresponding steady Ricci soliton of Chapter 1 is in fact the unique blow-up limit at the tip of M_k, then these singularities are necessarily accompanied by a Type I singularity modeled on $S^3/\mathbb{Z}_k$, because these solitons are asymptotically cylindrical.

Another interesting question is the following:

Question 1. *Can the Eguchi-Hanson singularity occur on a closed four dimensional manifold?*

The author conjectures that the answer is yes, however only non-generically. The simplest model on which to investigate this question is $M = M_2 \#_{\mathbb{R}P^3} M_2 \cong S^2 \times S^2$ equipped with an $U(2)$-invariant metric. One could carry out a construction as follows: Vary between an initial metric that encounters a $\mathbb{R} \times \mathbb{R}P^3$ neckpinch singularity and an initial metric that leads to the collapse of one of the S^2 factors of M. On the path between these two metrics there should be a metric whose Ricci flow evolution forms an Eguchi-Hanson singularity in finite time.

We have not touched upon the behavior of a general non-$U(2)$-invariant metric on TS^2. A first question would be:

Question 2. *Does the picture of Theorem 2.12.1 also hold for Ricci flows starting from non-$U(2)$-invariant perturbations of asymptotically cylindrical $U(2)$-invariant metrics on TS^2?*

And a final big question mark is the following:

Question 3. *Are there other four dimensional Ricci flat ALE spaces that can occur as blow-up limits in Ricci flow?*

So far all known Ricci flat ALE spaces in four dimensions are hyperkähler and it is not known whether non-hyperkähler examples exist. Kronheimer classified all hyperkähler ALE spaces [KronI89], [KronII89]. These spaces have one end that is asymptotic to the cone $\mathbb{R}^4/\Gamma$, where $\Gamma \subset U(2)$ is a certain finite group — a binary dihedral, tetrahedral, octahedral or icosahedral group. In the case that $\Gamma = \mathbb{Z}_k$ is cyclic, Gibbons and Hawking [GH78], [GH79] discovered a closed form $(3k-6)$-parameter family of such metrics. In the physics literature these metrics are known as multi-center Eguchi Hanson spaces. It would be interesting to see whether our results can be generalized to prove the existence of singularities modeled on these spaces.

2.2 Preliminaries

Notation

Here we collect some of the notation used throughout chapter 2.

- M_k, $k \in \mathbb{N}$: a manifold diffeomorphic to the blow-up of $\mathbb{C}^2/\mathbb{Z}_k$, $k \geq 1$, at the origin.
- S^2_o: the two-sphere added during the blow-up of $\mathbb{C}^2/\mathbb{Z}_k$.
- ξ: the radial coordinate on M_k or the parametrization of the $\mathbb{R}_{>0}$ factor in the product $\mathbb{R}_{>0} \times S^3/\mathbb{Z}_k \subset M_k$.
- o: a fixed point on S^2_o.
- Σ_p: denotes the orbit of p under the $U(2)$-action. For instance if $p \in S^2_o \subset M_k$ we have $\Sigma_p = S^2_o$ and when $p \in M_k \setminus S^2_o$ we have $\Sigma_p \cong S^3/\mathbb{Z}_k$.
- s: the geodesic distance from S^2_o, and often considered as a function of ξ and t.
- origin: refers to the point o.
- g: a metric of the form (2.2.2) or (2.2.3) unless otherwise stated
- d_g: the metric distance induced by g.
- $g(t)$: a time dependent family of metrics of the form (2.2.2) or (2.2.3).
- u, a, b: the warping functions of the metric (2.2.2). Depending on context these will be considered as functions of (ξ, t), (s, t) or (p, t), where p is a point on M_k.
- $Q \coloneqq \frac{a}{b}$.
- $B_g(p, r)$: the ball centered at p of radius r with respect to the metric g.

- $C_g(p,r)$, $r>0$: the subset of a cohomogeneity one $U(2)$-invariant Riemannian manifold (M,g) defined by
$$C_g(p,r)=\left\{q\in M \,\middle|\, d_g(q,\Sigma_p)<r\right\}.$$
The set $C_g(p,r)$ is diffeomorphic to either M_k or $\mathbb{R}\times S^3/\mathbb{Z}_k$.
- $\overline{C}_g(p,r)$: the closure of $C_g(p,r)$.
- T_{sing}: the singular time of a Ricci flow.
- $C^\infty_{U(2)}(M_k\times[0,T])$: The space of smooth $U(2)$-invariant functions $u:M_k\times[0,T]\to\mathbb{R}$.
- x,y: Kähler quantities introduced in section 2.4.

The manifold and metric

For $k\in\mathbb{N}$ let M_k be diffeomorphic to the blow-up of $\mathbb{C}^2/\mathbb{Z}_k$ at the origin. Denote by S^2_o the embedded two-sphere in M_k stemming from the blow-up, and fix some point o for 'origin' on S^2_o.

We now describe the $U(2)$-invariant metrics on M_k, $k\geq 1$, studied in this chapter. Let z_1,z_2 be the standard coordinates on $\mathbb{C}^2$ and let $U(2)$ act on $\mathbb{C}^2$ by left multiplication. This action descends to M_k, $k\in\mathbb{N}$. Note that M_k can be seen as the total space of the complex line bundle $O(-k)$ via

$$\begin{aligned}\pi: M_k &\longrightarrow S^2_o\\ (z_1,z_2)&\mapsto[z_1,z_2]\end{aligned}$$

Then $U(2)\cong U(1)\times SU(2)$ acts on M_k in the following way: The action of $U(1)$ rotates the fibres of π and $SU(2)$ acts on the base S^2_o via rotations. Now introduce the Hopf coordinates

$$\begin{aligned}z_1&=\xi\sin\eta\, e^{i(\psi+\phi)}=x_1+iy_1\\ z_2&=\xi\cos\eta\, e^{i(\psi-\phi)}=x_2+iy_2\end{aligned}$$

on $\mathbb{C}^2_*$, where $\xi>0$, $\eta\in[0,\pi/2]$ and $\psi,\phi\in[0,2\pi)$. These coordinates descend to M_k. In particular, this allows us to endow M_k with the radial coordinate $\xi: M_k\to\mathbb{R}_{\geq 0}$, by continuously extending ξ to S^2_o by taking $\xi=0$ on S^2_o. Note that the coordinate ξ is only smooth on $M_k\setminus S^2_o$.

A computation shows that the standard euclidean metric

$$g_{eucl}=dx_1^2+dy_1^2+dx_2^2+dy_2^2$$

may be written as

$$g_{eucl}=d\xi^2+\xi^2\left(d\eta^2+\sin^2(2\eta)d\phi^2+[d\psi-\cos(2\eta)d\phi]^2\right)$$

in Hopf coordinates. The 1-form

$$\omega := d\psi - \cos(2\eta)d\phi$$

is dual to the Hopf fibre directions, or equivalently dual to the vector field generated by the $U(1)$ action. Furthermore

$$d\eta^2 + \sin^2(2\eta)d\phi^2 \tag{2.2.1}$$

is the pull-back of the Fubini-Study metric g_{FS} on $\mathbb{C}P^1$, normalized to have constant sectional curvature equal to $\frac{1}{4}$.

From the above we see that the warped-product metric

$$g = u(\xi)^2 d\xi^2 + a(\xi)^2 \omega \otimes \omega + b(\xi)^2 \pi^*(g_{FS}) \tag{2.2.2}$$

is the most general $U(2)$-invariant metric on $\mathbb{C}^2_*$ and descends to a $U(2)$-invariant metric on the open dense set $\mathbb{C}^2_*/\mathbb{Z}_k \subset M_k$. It will be useful to introduce the change of coordinates defined by

$$ds = u(\xi)d\xi$$

and $s = 0$ at $\xi = 0$. Then for $p \in M_k$ the quantity $s(p) = d_g(p, S^2_o)$ describes the radial distance of p from S^2_o. In these coordinates the metric becomes

$$g = ds^2 + a(s)^2 \omega \otimes \omega + b(s)^2 \pi^*(g_{FS}), \tag{2.2.3}$$

where in a slight abuse of notation we consider a and b as functions of s. Depending on the context we will consider a and b either as functions of s or ξ.

The metric g can be extended to a metric on all of M_k by taking $a(0) = 0$ and $b(0) > 0$. In other words we shrink the Hopf fibre directions to zero as $s \to 0$ or equivalently as we approach S^2_o. Note that for every $p \in S^2_0$

$$ds^2 + a(s)^2 \omega \otimes \omega$$

is the pull-back of g onto the fibre $\pi^{-1}(p)$. As $U(1)$ acts on the fibre $\pi^{-1}(p)$, we see that $\pi^{-1}(p)$ is a union of S^1 orbits and p. Furthermore, such a S^1 orbit in $\pi^{-1}(p) \subset M_k$ is parametrized by $0 \leq \psi < \frac{2\pi}{k}$ and, by the form of the metric (2.2.3), such an S^1 orbit at radial distance s from S^2_o has a circumference of length $\frac{2\pi}{k}a(s)$. Because

$$\frac{2\pi}{k}a(s) = \frac{2\pi}{k}a_s(0)s + O(s^2) \text{ as } s \to 0$$

we must require that $a_s(0) = k$ in order to avoid a conical singularity at S^2_o. This is how the topology of the manifold enters the analysis of the Ricci flow equation. Additionally requiring that $a(s)$ and $b(s)$ can be extended to an odd and even function, respectively, around $s = 0$ is a sufficient condition for the metric g to be smoothly extendable to all of M_k [VZ18]. In the rest of Chapter 2 all metrics considered will be of the form (2.2.2) or

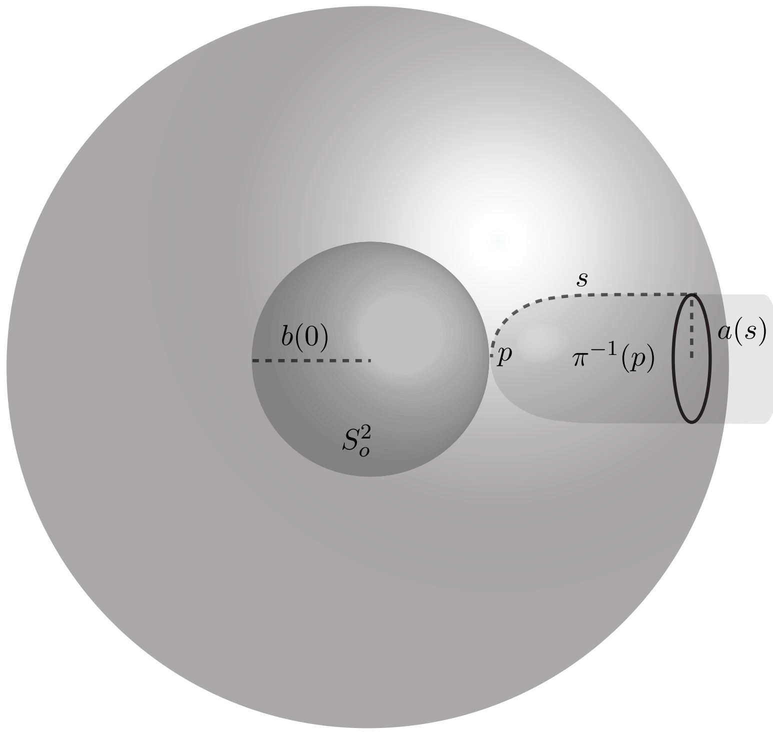

Figure 2.1: Diagram of the manifold M_k close to the tip

equivalently (2.2.3). In Figure 2.1 the manifold M_k and its metric close to the two sphere S_o^2 is schematically depicted.

(M_k, g), $k \in \mathbb{N}$, are cohomogeneity one manifolds, meaning that the generic orbits of the $U(2)$ action are of codimension 1. The generic orbit is also called the principal orbit. The non-generic orbits are called non-principal orbits. In the case of M_k the principal orbits are diffeomorphic to $S^3/\mathbb{Z}_k$ and the single non-principal orbit is S_o^2 and of codimension 2. Below we introduce some notation that we frequently employ:

Definition 2.2.1. Assume (M, g) is a $U(2)$-invariant cohomogeneity one manifold with principal orbit $S^3/\mathbb{Z}_k$ for some fixed $k \in \mathbb{N}$ and g is a metric of the form (2.2.3). Let $p \in M$ and $r > 0$. Then

- Let $\Sigma_p \subset M$ denote the orbit of p under the $U(2)$-action.
- Let Σ_p^+ be the set of all points $q \in M$ that can be joined via path τ to p with $g\left(\dot{\tau}, \frac{\partial}{\partial s}\right) \geq 0$.

- Let
$$C_g(p,r) \coloneqq \left\{ q \in M \,\middle|\, d_g(q,\Sigma_p) < r \right\}$$
- Let
$$C_g^+(p,r) \coloneqq \left\{ q \in \Sigma_p^+ \,\middle|\, d_g(q,\Sigma_p) < r \right\}$$

Note that we have $\Sigma_p \cong S^3/\mathbb{Z}_k$ if p lies on a principal orbit and $\Sigma_p \cong S^2$ if p lies on a non-principal orbit.

The connection, Laplacian and curvature tensor

We now compute the connection, Laplacian and curvature tensor for metrics of the form (2.2.3). To obtain the corresponding expressions for the metric (2.2.2) use the relation
$$\frac{\partial}{\partial s} = \frac{1}{u}\frac{\partial}{\partial \xi}.$$
Take the orthonormal basis
$$e^0 = ds \qquad e^1 = a\,[d\psi - \cos(2\eta)d\phi] \qquad e^2 = bd\eta \qquad e^3 = b\sin(2\eta)d\phi$$
of T^*M. Let e_i, $i = 1,2,3,4$, be the corresponding dual basis of T_*M. Define the connection 1-forms θ_i^j by $\nabla e_i = \theta_i^j e_j$ and the curvature 2-forms Ω_i^j by $R(\cdot,\cdot)e_i = \Omega_i^j e_j$. With help of Cartan's structure equations
$$\begin{aligned}
\theta_i^j &= -\theta_j^i \\
de^i &= -\theta_j^i \wedge e^j \\
\Omega_i^j &= d\theta_i^j + \theta_k^j \wedge \theta_i^k
\end{aligned}$$
one can compute the connection 1-forms and curvature 2-forms. First note that
$$\begin{aligned}
de^0 &= 0 \\
de^1 &= \frac{a_s}{a} e^0 \wedge e^1 + \frac{2a}{b^2} e^2 \wedge e^3 \\
de^2 &= \frac{b_s}{b} e^0 \wedge e^2 \\
de^3 &= \frac{b_s}{b} e^0 \wedge e^3 + \frac{2}{b}\cot(2\eta) e^2 \wedge e^3.
\end{aligned}$$
Hence we obtain the connection 1-forms θ_j^i:
$$\begin{aligned}
\theta_0^1 &= \frac{a_s}{a} e^1 & \qquad \theta_2^1 &= \frac{a}{b^2} e^3 \\
\theta_0^2 &= \frac{b_s}{b} e^2 & \qquad \theta_3^2 &= -\frac{a}{b^2} e^1 - \frac{2}{b}\cot(2\eta) e^3 \\
\theta_0^3 &= \frac{b_s}{b} e^3 & \qquad \theta_1^3 &= \frac{a}{b^2} e^2
\end{aligned}$$

Therefore

$$\nabla_{e_0} e_0 = 0 \qquad \nabla_{e_1} e_1 = -\frac{a_s}{a} e_0 \qquad \nabla_{e_2} e_2 = -\frac{b_s}{b} e_0 \qquad \nabla_{e_3} e_3 = -\frac{b_s}{b} e_0 - \frac{2}{b}\cot(2\eta) e_2$$

from which we can derive the expression for the Laplacian of a $U(2)$-invariant function $f : M_k \to \mathbb{R}$:

$$\Delta f = \sum_{i=0}^{3} \nabla^2_{e_i, e_i} f = f_{ss} + \left(\frac{a_s}{a} + 2\frac{b_s}{b}\right) f_s. \tag{2.2.4}$$

Finally, we may compute the components

$$R_{ijkl} = g\left(R(e_k, e_l) e_j, e_i\right).$$

of the curvature tensor. Below we list its non-zero components

$$\begin{aligned}
R_{0101} &= -\frac{a_{ss}}{a} = K_1 \\
R_{0202} &= -\frac{b_{ss}}{b} = K_2 \\
R_{0303} &= -\frac{b_{ss}}{b} = K_3 \\
R_{0123} &= -\frac{2}{b^2}\left(a_s - Q b_s\right) = M_1 \\
R_{0231} &= \frac{1}{b^2}\left(a_s - Q b_s\right) = M_2 \\
R_{0312} &= \frac{1}{b^2}\left(a_s - Q b_s\right) = M_3 \\
R_{1212} &= \frac{a^2}{b^4} - \frac{a_s b_s}{ab} = H_{12} \\
R_{2323} &= \frac{4}{b^2} - 3\frac{a^2}{b^4} - \left(\frac{b_s}{b}\right)^2 = H_{23} \\
R_{3131} &= \frac{a^2}{b^4} - \frac{a_s b_s}{ab} = H_{31}.
\end{aligned}$$

All other components are either determined by the standard symmetries of the curvature tensor or are zero.

The Ricci flow equation

With help of the above list of curvature components one can check that the Ricci tensor is diagonal and hence the form of the metric (2.2.2) is preserved by Ricci flow. Allowing the warping functions a, b and p to vary in time, the Ricci flow equation (2.1.1) in (ξ, t)

coordinates can be expressed as a system of coupled parabolic equations in a, b and u.

$$\partial_t u = \frac{1}{a}\partial_\xi\left(\frac{a_\xi}{u}\right) + \frac{2}{b}\partial_\xi\left(\frac{b_\xi}{u}\right) \tag{2.2.5}$$

$$\partial_t a = \frac{1}{u}\partial_\xi\left(\frac{a_\xi}{u}\right) - 2\frac{a^3}{b^4} + 2\frac{a_\xi b_\xi}{bu^2} \tag{2.2.6}$$

$$\partial_t b = \frac{1}{u}\partial_\xi\left(\frac{b_\xi}{u}\right) - \frac{4}{b} + 2\frac{a^2}{b^3} + \frac{a_\xi b_\xi}{au^2} + \frac{b_\xi^2}{bu^2} \tag{2.2.7}$$

Define the time dependent radial distance function $s = s(\xi, t)$ by

$$ds = u(\xi, t)d\xi$$

Then

$$s(\xi, t) = \int_0^\xi u(\xi, t)\,\mathrm{d}\xi \tag{2.2.8}$$

and

$$\frac{\partial}{\partial s} = \frac{1}{u}\frac{\partial}{\partial \xi}. \tag{2.2.9}$$

Furthermore the commutation relation

$$[\partial_t, \partial_s] = -\frac{\partial_t u}{u}\partial_s$$

holds. In terms of s we can use (2.2.9) to rewrite the Ricci flow equation in a slightly simpler form

$$\frac{\partial_t u}{u} = \frac{a_{ss}}{a} + 2\frac{b_{ss}}{b} \tag{2.2.10}$$

$$\partial_t a = a_{ss} - 2\frac{a^3}{b^4} + 2\frac{a_s b_s}{b} \tag{2.2.11}$$

$$\partial_t b = b_{ss} - \frac{4}{b} + 2\frac{a^2}{b^3} + \frac{a_s b_s}{a} + \frac{b_s^2}{b}. \tag{2.2.12}$$

Note that the dependence of the right hand side of this system of equations on ξ is hidden in the variable $s = s(\xi, t)$. However we can write the equations in terms of (s, t) by introducing the functions

$$\tilde{a}(s, t) = a(\xi, t)$$
$$\tilde{b}(s, t) = b(\xi, t)$$

and noting that

$$\partial_t a\big|_\xi = \partial_t \tilde{a}\big|_s + \partial_s \tilde{a}\big|_t \frac{\partial s}{\partial t}\big|_\xi$$
$$\partial_t b\big|_\xi = \partial_t \tilde{b}\big|_s + \partial_s \tilde{b}\big|_t \frac{\partial s}{\partial t}\big|_\xi.$$

By slight abuse of notation, however, we will drop the tilde and consider the warping functions a,b and u as functions of either (p,t), $p \in M_k$ or (ξ,t) or (s,t), depending on context. In (s,t) coordinates the Ricci flow equation reads

$$\partial_t a\big|_s = a_{ss} - 2\frac{a^3}{b^4} + 2\frac{a_s b_s}{b} - a_s \frac{\partial s}{\partial t} \tag{2.2.13}$$

$$\partial_t b\big|_s = b_{ss} - \frac{4}{b} + 2\frac{a^2}{b^3} + \frac{a_s b_s}{a} + \frac{b_s^2}{b} - b_s \frac{\partial s}{\partial t} \tag{2.2.14}$$

where

$$\frac{\partial s}{\partial t}\big|_\xi = \int_0^s \frac{a_{ss}}{a} + 2\frac{b_{ss}}{b}\,\mathrm{d}s \tag{2.2.15}$$

Whenever we differentiate a function $f : M_k \times [0,T]$ with respect to time, unless stated otherwise, assume that the point on the manifold M_k is held fixed. If we differentiate with respect to time while holding s fixed we will denote the partial derivative by $\partial_t f|_s$ to avoid confusion. Because s is a function of (ξ,t), in general for fixed $s_0 > 0$ the set $\{s = s_0\} \subset M_k$ is dependent on time. Therefore holding s or ξ fixed during partial differentiation produces very different results.

This following property of the warping functions a and b will be used throughout Chapter 2.

Lemma 2.2.2. *Let $(M_k, g(t))$, $t \in [0,T)$, be a smooth Ricci flow solution. Then for all $t \in [0,T]$ the warping functions $a(s,t)$ and $b(s,t)$ can be extended to an odd and even function, respectively, on $\mathbb{R}$.*

Proof. Note that a necessary condition for a metric g of the form (2.2.3) to be smooth is that its corresponding warping functions a and b are extendable to odd and even functions, respectively, on $\mathbb{R}$. Therefore the desired result follows. Alternatively, notice that if the warping functions of the initial data $a(s,0)$ and $b(s,0)$ can be extended to an odd and even function, respectively, on $\mathbb{R}$, we can also extend the equations (2.2.13), (2.2.14) and (2.2.15) to all of $\mathbb{R}$. An inspection of these equations shows that the parity of a and b is preserved under the flow. $\square$

Recap of blow-up limits of singularities

As mentioned above, every complete Riemannian manifold (M,g) of bounded curvature admits a short-time Ricci flow starting from g, however singularities may develop in finite time. Similar to the study of other nonlinear equations, it is very useful to consider blow-up limits of singularities. We briefly sketch the idea here: Assume $(M, g(t))$, $t \in [0, T_{sing})$, is a Ricci flow encountering a curvature singularity as $t \to T_{sing}$. Let (p_i, t_i) with $p_i \in M$ and $t_i \to T_{sing}$ be a sequence of points in spacetime such that

$$K_i := |Rm_{g(t_i)}|_{g(t_i)}(p_i) = \sup_{t \le t_i} |Rm_{g(t)}|_{g(t)}$$

and

$$K_i \to \infty \ \text{ as } \ i \to \infty.$$

Take the rescaled metrics

$$g_i(t) = K_i g\left(t_i + \frac{t}{K_i}\right), \quad t \in [-K_i t_i, 0].$$

Then $(M_k, g_i(t), p_i)$ subsequentially converge, in the Cheeger-Gromov sense, to a pointed ancient Ricci flow solution $(M_\infty, g_\infty(t), p_\infty), t \in (-\infty, 0]$ (see [ChI, Theorem 6.68] for more details). Note that in general $M_\infty \neq M$. A Ricci flow is called ancient if it can be extended to a time interval of the form $(-\infty, T)$, $T \in \mathbb{R}$. The blow-up limit $(M_\infty, g_\infty(t), p_\infty)$ is called the singularity model and yields important geometrical information on the shape of the singularity. Hamilton [Ham95] distinguishes between Type I and II singularities, depending on the rate of curvature blow-up, i.e. for Type I

$$\sup_{M \times [0,T)} (T_{sing} - t)\, |Rm_{g(t)}|_{g(t)} < \infty$$

and for Type II

$$\sup_{M \times [0,T)} (T_{sing} - t)\, |Rm_{g(t)}|_{g(t)} = \infty.$$

By the work of Naber [N10], and Enders, Müller and Topping [EMT11] it is known that Type I singularities are modeled on shrinking Ricci solitons. One hopes — although it has not been proven — that all Type II singularities are modeled on steady solitons, as to date all known examples are.

2.3 The maximum principle

Assume we are given a Ricci flow $(M_k, g(t))$, $t \in [0, T]$, $k \geq 1$. We make the following definition:

Definition 2.3.1. Let $C^\infty_{U(2)}(M_k \times [0, T])$ be the space of smooth $U(2)$-invariant functions

$$u : M_k \times [0, T] \to \mathbb{R}.$$

In this section we prove a maximum principle for operators

$$P : C^\infty_{U(2)}(M_k \times [0, T]) \to C^\infty_{U(2)}(M_k \times [0, T])$$

that in (ξ, t) coordinates and away from the non-principal orbit S^2_o of M_k can be written in the form

$$P[u] = \partial_{ss} u + \left(m\frac{a_s}{a} + n\frac{b_s}{b}\right) u_s + cu - \partial_t u, \quad m, n \in \mathbb{R}, \tag{2.3.1}$$

where $c \in C^\infty_{U(2)}(M_k \times [0,T])$. Recall from section 2.2 that we are interpreting the s derivative as

$$\frac{\partial}{\partial s} = \frac{1}{u}\frac{\partial}{\partial \xi}.$$

It is useful to work in (s,t) coordinates, in which case the operator $P[u]$ can be expressed as

$$P[u] = \partial_{ss}u + \left(m\frac{a_s}{a} + n\frac{b_s}{b} - \frac{\partial s}{\partial t}\right)u_s + cu - \partial_t\Big|_s u, \tag{2.3.2}$$

where we recall the expression (2.2.15) for $\frac{\partial s}{\partial t}$. Note $P[u]$ is degenerate at the origin $s=0$ as

$$m\frac{a_s}{a} + n\frac{b_s}{b} - \frac{\partial s}{\partial t} \sim \frac{m}{s} \text{ for } 0 < |s| \ll 1.$$

However, in (s,t) coordinates the smoothness of u and c is equivalent to saying that $u(s,t), c(s,t)$ can be extended to smooth even functions around $s=0$ by defining $u(s,t) = u(-s,t)$ and $v(s,t) = v(-s,t)$ for $s \leq 0$. Hence we see that $u_s = c_s = 0$ at $s = 0$ and via L'Hôpital's Rule we obtain the following representation of $P[u]$ on the principal orbit S^2_0:

$$P[u] = (m+1)\partial_{ss}u + cu - \partial_t u.$$

The maximum principle derived for P below depends on the sign of $(m+1)$:

Theorem 2.3.2. *Let $(M_k, g(t))$, $k \in \mathbb{N}$, $t \in [0,T]$ be a Ricci flow with bounded curvature. Let P be an operator of the form (2.3.1) and $u \in C^\infty_{U(2)}(M_k \times [0,T])$. If*

$$P[u] \leq 0 \;\text{ in }\; M_k \times [0,T]$$

and there exist constants $M, \sigma > 0$ such that the growth conditions

$$u(s,t) \geq -M\exp(-\sigma s^2)$$
$$c(s,t) \leq M\left(|s|^2 + 1\right)$$

are satisfied, then the following holds true:

Case 1 *$(1+m \leq 0)$ If*

$$u(s,0) \geq 0 \;\text{ for }\; s \geq 0$$
$$u(0,t) \geq 0 \;\text{ for }\; t \in [0,T]$$

then $u(s,t) \geq 0$ on $[0,\infty) \times [0,T]$. Furthermore, if $u = 0$ somewhere on $(0,\infty)\times(0,T]$ then $u = 0$ everywhere.

Case 2 *$(1+m > 0)$ If*

$$u(s,0) \geq 0 \;\text{ for }\; s \geq 0$$

then $u(s,t) \geq 0$ on $[0,\infty) \times [0,T]$. Furthermore, if $u = 0$ somewhere on $[0,\infty)\times(0,T]$ then $u = 0$ everywhere.

Before proving Theorem 2.3.2 we need to derive some bounds on $\frac{a_s}{a}$, $\frac{b_s}{b}$ and $\frac{1}{b^2}$ for metrics g of bounded curvature. This will allow us to bound the coefficients appearing in the expression (2.3.2) of the operator $P[u]$.

Lemma 2.3.3. *Let (M_k, g), $k \in \mathbb{N}$, and $K > 0$ such that $|Rm_g|_g \leq K$ on M_k. Then everywhere on M_k we have*

1. $b^2 \geq \frac{1}{K}$
2. $\left(\frac{b_s}{b}\right)^2 \leq 5K$
3. $\frac{Q^2}{b^2} \leq \frac{5}{3}K$

Proof. From the curvature components derived in subsection 2.2 we see that

$$b^2 H_{23} = 4 - 3Q^2 - b_s^2 \tag{2.3.3}$$

$$b^2 H_{12} = Q^2 - \frac{a_s b_s}{Q} \tag{2.3.4}$$

At a local minimum $b_s = 0$ we thus have

$$b^2 = \frac{4}{3H_{12} + H_{23}} \geq \frac{1}{K}$$

Now we argue by contradiction. Assume that there exists a $s^* > 0$ and $\delta > 0$ such that at $s = s^*$ we have $b^2 < \frac{1-\delta}{K}$. From above it follows that $b_s < 0$ for $s \geq s^*$ and hence, because $b > 0$ everywhere, $\lim_{s\to\infty} b_s = 0$. Equation (2.3.3) then shows that

$$\begin{aligned} Q^2 &= \frac{1}{3}\left(4 - b^2 H_{23} - b_s^2\right) \\ &\geq 1 + \frac{\delta}{3} - \frac{b_s^2}{3} \\ &\geq 1 + \frac{\delta}{4} \end{aligned}$$

for s sufficiently large. Then (2.3.4) implies that eventually

$$a_s \frac{b_s}{Q} \geq \frac{5}{4}\delta.$$

Dividing by $\frac{b_s}{Q}$ shows that

$$a_s \to -\infty \quad \text{as} \quad s \to \infty$$

contradicting $a \geq 0$. This proves the first bound. To prove the second bound note that

$$\left(\frac{b_s}{b}\right)^2 = \frac{4 - 3Q^2}{b^2} - H_{23} \leq \frac{4}{b^2} + K \leq 5K,$$

where the last inequality follows from (1). For the third bound we have

$$\frac{Q^2}{b^2} = \frac{1}{3}\left(\frac{4}{b^2} - \left(\frac{b_s}{b}\right)^2 - H_{23}\right) \leq \frac{5}{3}K.$$

□

Lemma 2.3.4. *Let (M_k, g), $k \in \mathbb{N}$, and $K > 0$ such that $|Rm_g|_g \leq K$ on M_k. Then everywhere on M_k we have*

$$-2\sqrt{K} < \frac{a_s}{a} < \frac{1}{s} + \sqrt{K} \tag{2.3.5}$$

Proof. The quantity $\phi = \frac{a_s}{a}s$ satisfies the ODE

$$\frac{d\phi}{ds} = s\frac{a_{ss}}{a} + \frac{\phi(1-\phi)}{s}$$

and by L'Hôpital's rule we have $\phi(0) = 1$. Note that the function ϕ can be extended to an even function on $\mathbb{R}$. Therefore $\frac{d\phi}{ds}(0) = 0$ and there exists a small $\epsilon > 0$ such that

$$\phi \leq 1 + \sqrt{K}s \text{ for } 0 \leq s \leq \epsilon.$$

Actually the inequality holds for all $s \geq 0$, since whenever $\phi(s) = 1 + \sqrt{K}s$ we have

$$\frac{d\phi}{ds} = s\left(\frac{a_{ss}}{a} - K\right) - \sqrt{K} < 0,$$

since $|\frac{a_{ss}}{a}| = |R_{0101}| \leq K$. this proves the the upper bound of (2.3.5).

To prove the lower bound assume that $a_s(s_0) < 0$. For every $s_1 > s_0$ there exists a $s^* \in (s_0, s_1)$ such that

$$a_s(s_1) - a_s(s_0) = (s_1 - s_0)a_{ss}(s^*) \leq (s_1 - s_0)K|a(s^*)|$$

by the mean value theorem. It follows that

$$a_s(s) \leq \frac{1}{2}a_s(s_0) \text{ for } s_0 \leq s \leq s_0 + \frac{1}{2K}\left|\frac{a_s(s_0)}{a(s_0)}\right|.$$

Therefore

$$0 \leq a\left(s_0 + \frac{1}{2K}\left|\frac{a_s(s_0)}{a(s_0)}\right|\right) \leq a(s_0) + \frac{1}{2K}\left|\frac{a_s(s_0)}{a(s_0)}\right|\frac{a_s(s_0)}{2}$$

which implies

$$\frac{a_s(s_0)}{a(s_0)} \geq -2\sqrt{K}.$$

This concludes the proof. □

Now we may proceed to proving the maximum principle of Theorem 2.3.2.

Proof of Case 1 of Theorem 2.3.2. Let $K > 0$ such that

$$\sup_{M_k \times [0,T]} |Rm_{g(t)}|_{g(t)} \leq K.$$

Introduce the new variable $r := r(s,t)$ defined by

$$r(r+2) = s^2$$

and let

$$\overline{u}(r,t) = u(\sqrt{r(r+2)}, t).$$

Note that $2r \sim s^2$ for $s \ll 1$ and $r \sim s$ for $s \gg 1$. We make this substitution to remove the apparent singularity at $s = 0$ in the (s,t) coordinate representation (2.3.2) of the operator $P[u]$. The function $\overline{u}$ is smooth, because u is extendable to an even function around the origin (see [W43]). Rewriting (2.3.2) in terms of (r,t) coordinates we see that $\overline{u}$ satisfies the inequality

$$\partial_t \overline{u}\Big|_r \geq A(r)\partial_{rr}\overline{u} + B(r,t)\partial_r \overline{u} + C(r,t)\overline{u}$$

where A, B and C are smooth functions defined by

$$\begin{aligned} A(r) &= \frac{r(r+2)}{(r+1)^2} \\ B(r,t) &= \left(m\frac{a_s}{a} + n\frac{b_s}{b} - \frac{\partial s}{\partial t} \right) \frac{s}{r+1} + \frac{1}{(r+1)^3} \\ C(r,t) &= c(s,t) \end{aligned}$$

Note that above we regard s as a function of r. Recall that by Lemma 2.2.2 the functions a and b can be extended to an odd and even function, respectively, around the origin. Therefore the quantity

$$\left(m\frac{a_s}{a} + n\frac{b_s}{b} - \frac{\partial s}{\partial t} \right) s, \tag{2.3.6}$$

considered as a function of s, can be extended to an even function around the origin by [W43]. Hence this expression depends smoothly on r, showing that $B(r,t)$ is smooth. Similarly, we see that $C(r,t)$ is smooth. From the expression (2.2.15) for $\frac{\partial s}{\partial t}$ and the curvature components listed in subsection 2.2 it follows that

$$\left|\frac{\partial s}{\partial t}\right| = \left| \int_0^s -K_1 - 2K_2 \,\mathrm{d}s \right| \leq 3Ks \tag{2.3.7}$$

By Lemma 2.3.3 and Lemma 2.3.4 we hence see that

$$|B(r,t)| \leq M(r+1)$$

for some some positive constant M depending on K. Finally, noting that $A(r)$ is bounded and positive for $r > 0$, we can apply [F64, Theorem 9, p.43] to deduce that the weak maximum principle holds. Note that for any compact $U \subset M_k \times [0, T]$ we may assume that $c < 0$ on U by performing the transformation $\overline{u} \leftarrow \overline{u}e^{-\gamma t}$, for $\gamma = \gamma(U)$ chosen sufficiently large. Therefore the strong maximum principle follows from a slight adaptation of [F64, Theorem 1, p.34]. □

Proof of Case 2 of Theorem 2.3.2. We first prove the weak maximum principle. Taking $u' = ue^{-\gamma t}$ we see that u' satisfies

$$\partial_t u' \geq \partial_{ss} u' + \left(m\frac{a_s}{a} + n\frac{b_s}{b} \right) u'_s + (c - \gamma)u'.$$

As c is a smooth function of (s, t), we can choose γ sufficiently large such that in a neighborhood of $\{s = 0\} \times [0, T]$ we have $c - \gamma < 0$. Since $m + 1 > 0$ we see that u' cannot attain a negative minimum on $\{s = 0\} \times (0, T]$, as otherwise

$$0 \leq \partial_t u' = (1 + m)u'_{ss} + cu' > 0,$$

which is a contradiction. The weak maximum principle now follows by the proof of [F64, Theorem 9, p.43].

We only apply the strong maximum principle for $m \in \mathbb{N}$ and therefore only prove this case here. For the general case refer to [Fee13, Theorem 5.17]. Given a Ricci flow $(M_k, g(t))$, $t \in [0, T]$, define the corresponding family of rotationally symmetric spaces $(\mathbb{R}^{m+1}, h(t))$, $t \in [0, T]$, by

$$h = ds^2 + a^2(s, t) g_{S^m(\frac{1}{k})},$$

where $g_{S^m(\frac{1}{k})}$ is the round metric on S^m of sectional curvature k^2. A sufficient condition for h to be smooth at $s = 0$ is that a is extendable to an odd function around the origin and

$$a_s(0) = k.$$

Both these conditions are satisfied and we conclude that h is a smooth metric. The Laplacian of a rotationally symmetric function f on $(\mathbb{R}^{m+1}, h)$ is given by

$$\Delta_h f = f_{ss} + m\frac{a_s}{a} f_s$$

and thus the condition $P[u] \leq 0$ may be written as

$$\partial_t u \geq \Delta_{h(t)} u + n\frac{b_s}{b} u_s + cu$$

Note that for any bounded $U \subset M_k \times [0, T]$ we may assume that $c < 0$ on U by performing the transformation $u \leftarrow ue^{-\gamma t}$, for $\gamma = \gamma(U)$ chosen sufficiently large. Hence the desired result follows from [ChII, Theorem 12.40]. □

Remark 2.3.5. It was crucial in our analysis that u is extendable to an even function around the origin, as the following example demonstrates:

Consider the degenerate parabolic equation

$$u_t = u_{xx} - 2\frac{u_x}{x} + 2\frac{u}{x^2} \tag{2.3.8}$$

on $x, t \geq 0$ with initial data satisfying $u(x, 0) \leq 0$. If we take

$$u = xv$$

a computation shows that the above PDE corresponds to

$$v_t = v_{xx} \tag{2.3.9}$$

Now considering (2.3.9) as the heat equation on all of $\mathbb{R}$ we can set up initial data $v(x, 0)$ such that

$$v(x, 0) \leq 0 \ \text{ for } \ x \geq 0,$$

however the solution v to the heat equation becomes positive at some later time $t > 0$ and $x > 0$. This shows that $u \leq 0$ is not necessarily preserved by (2.3.8).

We also rely on a maximum principle for a system of parabolic inequalities on $u_1, u_2 \in C^\infty_{U(2)}(M_k \times [0, T])$ of the form

$$\partial_t u_1 \geq (u_1)_{ss} + \left(m\frac{a_s}{a} + n\frac{b_s}{b} \right) (u_1)_s + h_{11}u_1 + h_{12}u_2 \tag{2.3.10}$$

$$\partial_t u_2 \geq (u_2)_{ss} + \left(m\frac{a_s}{a} + n\frac{b_s}{b} \right) (u_2)_s + h_{21}u_1 + h_{22}u_2, \tag{2.3.11}$$

where $h_{ij} \in C^\infty_{U(2)}(M_k \times [0, T])$, $i, j = 1, 2$, are bounded and satisfy

$$h_{12}, h_{21} \geq 0 \ \text{ on } \ M_k \times [0, T].$$

We prove the following Lemma:

Lemma 2.3.6. *Let $(M_k, g(t))$, $t \in [0, T]$, be a Ricci flow with bounded curvature. Assume $u_1, u_2 \in C^\infty_{U(2)}(M_k \times [0, T])$ satisfy the above system (2.3.10)-(2.3.11) of parabolic inequalities and for some constants $M, \sigma > 0$*

$$u_1(s, t), u_2(s, t) \geq -M \exp(\sigma s^2) \ \textit{ for } t \in [0, T].$$

If

$$u_1(s, 0), u_2(s, 0) \geq 0 \ \textit{ for } s \geq 0$$
$$u_1(0, t), u_2(0, t) \geq 0 \ \textit{ for } t \in [0, T]$$

then $u_1, u_2 \geq 0$ on $M_k \times [0, T]$.

Proof. Writing the equation in terms of (r,t) as in the proof of Theorem 2.3.2 we obtain

$$\partial_t u_1\Big|_r \geq A(r)(u_1)_{rr} + B(r,t)(u_1)_r + H_{11}u_1 + H_{12}u_2$$
$$\partial_t u_2\Big|_r \geq A(r)(u_2)_{rr} + B(r,t)(u_2)_r + H_{21}u_1 + H_{22}u_2,$$

where $A(r)$, $B(r,t)$ are as in the proof of Theorem 2.3.2 and

$$H_{ij}(r,t) = h_{ij}(s(r),t), \quad i,j = 1,2.$$

After constructing a barrier function of the form

$$H(s,t) = \exp\left[\frac{k|s|^2}{1-\mu t} + \nu t\right], \quad 0 \leq t \leq \frac{1}{2\mu},$$

the result follows combining the arguments of [F64, Theorem 1, p.34] and [PW84, Theorem 13, p. 190]. □

2.4 Kähler quantities and the Eguchi-Hanson space

Recall that a complex structure J on a Riemannian manifold (M,g) satisfying

1. $g(V_1,V_2) = g(JV_1,JV_2)$ for all $V_1,V_2 \in TM$
2. $\nabla J = 0$

defines a Kähler structure. On the manifolds M_k, $k \geq 1$, we define two complex structures J_1 and J_2 by

$$J_1e_1 = e_0 \qquad J_1e_2 = e_3$$

and

$$J_2e_0 = e_2 \qquad J_2e_1 = e_3.$$

A computation shows that (M_k,g,J_1) is Kähler if and only if

$$b_s - Q = 0.$$

Similarly, (M_k,g,J_2) is Kähler if and only if

$$a_s + Q^2 - 2 = 0 \quad \text{and} \quad b_s - Q = 0.$$

Note that being Kähler with respect to J_2 automatically implies Kählerity with respect to J_1. This motivates the definition of the following *scale-invariant* quantities

$$x := a_s + Q^2 - 2$$
$$y := b_s - Q$$

to measure the deviation of a metric from being Kähler with respect to the complex structures J_1 and J_2. For example, the FIK shrinker [FIK03] is Kähler with respect to the complex structure J_1 and in our notation satisfies $y = 0$. The Eguchi-Hanson space is the unique Kähler manifold with respect to J_2 as the following lemma shows.

Lemma 2.4.1. *Amongst all Riemannian manifolds (M_k, g),$k \geq 1$, equipped with $U(2)$-invariant metric g of the form (2.2.3), up to scaling the Eguchi-Hanson space is the unique Kähler manifold with respect to the complex structure J_2. Furthermore being Kähler with respect to J_2 is equivalent to $x = y = 0$.*

Proof. By the above discussion being Kähler with respect to J_2 is equivalent to

$$x = y = 0. \tag{2.4.1}$$

Notice at $s = 0$ we have

$$x = a_s - 2 = 0$$

forcing the underlying manifold to be diffeomorphic to M_2 by the boundary conditions (2.1.3). Then in terms of a and b the condition $x = y = 0$ is equivalent to the first order system of equations

$$a_s = 2 - Q^2 \tag{2.4.2}$$

$$b_s = Q \tag{2.4.3}$$

Let a^E and b^E be a solution to this system of equations satisfying the initial conditions

$$a^E = 0$$
$$b^E = 1$$

at $s = 0$. Then by the scale-invariance of condition (2.4.1), for every $\lambda > 0$ the metric given by the warping functions $\lambda a^E(\lambda s)$ and $\lambda b^E(\lambda s)$ also satisfies (2.4.1). Hence up to rescaling there is a unique Kähler manifold with respect to the complex structure J_2. From [EH79] or [Cal79] we see that the metric given by a^E and b^E is homothetic to the Eguchi-Hanson metric. $\square$

In the rest of Chapter 2 we denote by g_E the Eguchi-Hanson metric with warping functions a^E and b^E normalized such that $b^E = 1$ on S^2_o. Note that the normalization condition is equivalent to saying that the area of the exceptional divisor S^2_o is equal to 2π.

Lemma 2.4.2. *The warping functions a^E and b^E of the Eguchi-Hanson metric satisfy the following properties*

$$\begin{aligned} a^E, b^E &\sim s \;\; as \;\; s \to \infty \\ a^E_{ss} &< 0 \;\; for \;\; s \geq 0 \\ b^E_{ss} &> 0 \;\; for \;\; s \geq 0 \\ \frac{a^E}{b^E} &< 1 \;\; for \;\; s \geq 0 \end{aligned}$$

Proof. For brevity write a and b for a^E and b^E, respectively. Note that on the Eguchi-Hanson background we have

$$Q_s = \frac{1}{b}\left(a_s - Qb_s\right) = \frac{2}{b}\left(1 - Q^2\right),$$

where the last equality follows from (2.4.2) and (2.4.3). As $Q = 0$ at $s = 0$ it follows that

$$Q < 1 \;\; \text{for} \;\; \geq 0$$

and hence

$$Q_s > 0 \;\; \text{for} \;\; s \geq 0.$$

As

$$a_s = Q_s b + Q b_s = 2 - Q^2$$

it follows that

$$a_{ss} = -2QQ_s < 0.$$

Similarly

$$b_{ss} = Q_s > 0.$$

Therefore the limits

$$a_\infty := \lim_{s \to \infty} a_s$$

and

$$b_\infty := \lim_{s \to \infty} b_s$$

both exist. From the system of differential equations (2.4.2) and (2.4.3) we then see that

$$a_\infty = b_\infty = 1.$$

This concludes the proof. $\square$

2.5 Some preserved conditions

In this section we derive various *scale-invariant* inequalities that are preserved by a Ricci flow $(M_k, g(t))$, $t \in [0, T]$, $k \in \mathbb{N}$. The scale-invariance is crucial, as it ensures that the inequalities pass to blow-up limits and therefore also constrain their geometry. The preserved inequalities we list in this section will play an important role in all subsequent sections.

Section Outline. A central quantity in our analysis is

$$Q = \frac{a}{b}.$$

In geometric terms, Q measures the deviation of the cross-sectional $S^3/\mathbb{Z}_k$ in M_k from being round. That is, when $Q = 1$ the cross-section is round and as $Q \to 0$ the cross-sectional $S^3/\mathbb{Z}_k$ collapses along the S^1 Hopf fibres to a two-sphere. A computation shows that the evolution equation of Q is

$$\partial_t Q = Q_{ss} + 3\frac{b_s}{b}Q_s + \frac{4}{b^2}Q(1 - Q^2). \tag{2.5.1}$$

Therefore one expects that the inequality $Q \leq 1$ is preserved by Ricci flow, which in Lemma 2.5.2 we prove to be the case.

Apart from Q, the Kähler quantities x and y introduced in section 2.4 are used throughout this chapter and are one of the key ingredients in showing that certain Ricci flows on M_2 develop singularities modeled on the Eguchi-Hanson space. We show in Lemma 2.5.3 and Lemma 2.5.5 that the inequalities

$$x \leq 0$$

and

$$y \leq 0$$

are both preserved. Furthermore, using the maximum principle for systems of weakly coupled parabolic equations of Lemma 2.3.6, we show in Lemma 2.5.6 that

$$a_s, b_s \geq 0$$

is preserved. In Lemma 2.5.7 we show that on a Ricci flow background satisfying $Q \leq 1$ and $y \leq 0$, for any $C > 2$ the inequality $a_s \leq C$ is preserved. In the following sections we will mainly consider Ricci flows satisfying $a_s, b_s \geq 0$, $y \leq 0$, $Q \leq 1$ and $a_s \leq C$. This gives us enough control on a and b to prove many interesting results.

Finally, we show that whenever a subset of the inequalities $Q \leq 1$, $y \leq 0$ and $a_s, b_s \geq 0$ hold, the details of which are discussed below, the following inequalities

$$\begin{aligned}
T_1 &= a_s + 2Q^2 - 2 \geq 0 \\
T_2 &= Qy - x = -a_s + Qb_s + 2\left(1 - Q^2\right) \geq 0 \\
T_3 &= a_s - Qb_s - Q^2 + 1 \geq 0 \\
\min(T_1, T_4) &\geq 0
\end{aligned}$$

where

$$T_4 = a_s - \frac{1}{2}Qb_s - \left(1 - Q^2\right)$$

are preserved by the Ricci flow. The precise statements and proofs of these preserved inequalities can be found in Lemmas 2.5.8, 2.5.9, 2.5.10 and 2.5.11 below.

The main idea in constructing the above inequalities is to study the evolution equation of the scale-invariant quantities

$$T_{(\alpha,\beta,\gamma)} = \alpha a_s + \beta Q b_s + \gamma Q^2, \quad \alpha, \beta, \gamma \in \mathbb{R}. \tag{2.5.2}$$

For this we need to compute the evolution equations of a_s, Qb_s and Q^2. Recall Definition 2.3.1 of $C^\infty_{U(2)}(M_k \times [0,T])$. To simplify the formulae slightly, define the linear operator

$$L : C^\infty_{U(2)}(M_k \times [0,T]) \to C^\infty_{U(2)}(M_k \times [0,T])$$

by

$$L[u] = u_{ss} + \left(2\frac{b_s}{b} - \frac{a_s}{a}\right) u_s$$

away from the non-principal orbit S^2_o. As in section 2.3 we may use L'Hôptital's rule to find a representation of L on the non-principal orbit S^2_o. Then, as we show in the Appendix A, the evolution equations of a_s, Qb_s and Q^2 can be written as

$$\partial_t a_s = L[a_s] + \frac{1}{b^2}\left(-2a_s b_s^2 - 6Q^2 a_s + 8Q^3 b_s\right) \tag{2.5.3}$$

$$\partial_t Q b_s = L[Qb_s] + \frac{1}{b^2}\left(4Q^2 a_s - 10Q^3 b_s - 2Qb_s^3 + 8Qb_s\right) \tag{2.5.4}$$

$$\partial_t Q^2 = L[Q^2] + \frac{1}{b^2}\left(4Qa_s b_s - 4Q^2 b_s^2 - 8Q^4 + 8Q^2\right). \tag{2.5.5}$$

Since the operator L is linear, one sees that $T_{(\alpha,\beta,\gamma)}$ satisfies an evolution equation of the form

$$\partial_t T_{(\alpha,\beta,\gamma)} = L[T_{(\alpha,\beta,\gamma)}] + \frac{1}{b^2}C_{(\alpha,\beta,\gamma)}, \tag{2.5.6}$$

where $C_{(\alpha,\beta,\gamma)}$ is a function of a_s, b_s and Q. This evolution equation is very useful, as it allows us to systematically search for preserved inequalities. In particular, if we can find $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ for which we can determine the sign of $C_{(\alpha,\beta,\gamma)}$ at a local extrema of $T_{(\alpha,\beta,\gamma)}$ at which $T_{(\alpha,\beta,\gamma)} = \delta$, it follows from the maximum principle of Theorem 2.3.2 that, depending on the sign, either

$$T_{(\alpha,\beta,\gamma)} \geq \delta$$

or

$$T_{(\alpha,\beta,\gamma)} \leq \delta$$

is a preserved inequality.

We searched for real numbers α, β, γ and δ leading to preserved conditions that yield the most useful control of the geometry of the flow. This is how we found the quantities T_1, T_2, T_3 and T_4. In later sections we will make heavy use of each of their respective inequalities. For instance, we use the preserved inequalities $T_1 \geq 0$ to exclude shrinking solitons on M_k, $k \geq 2$, in the next section. Finally, in section 2.11 we generalize the above idea to find a continuously varying family of conserved inequalities.

Statement and proof of results. In this subsection we list the precise statements and proofs of the results stated in the section outline. Before we begin, we prove the following technical lemma, which we need for verifying the growth conditions of the maximum principle of Theorem 2.7.5.

Lemma 2.5.1. *Let (M_k, g), $k \in \mathbb{N}$, satisfy $|Rm_g|_g \leq K$. Then*

$$|a_s|, |Qb_s|, |Q^2| = O(\exp(2\sqrt{K}s)).$$

Proof. By the curvature components listed in section 2.2 we see that

$$\left|\frac{a_{ss}}{a}\right|, \left|\frac{b_{ss}}{b}\right| \leq K.$$

Integrating

$$b_{ss} \leq bK,$$

shows that

$$b = O(\exp(\sqrt{K}s)).$$

From Lemma 2.3.3 we have

$$Q^2 \leq \frac{5}{3}Kb^2$$

from which we conclude that

$$Q^2 = O(\exp(2\sqrt{K}s)).$$

Similarly, Lemma 2.3.3 shows

$$|b_s| \leq \sqrt{5K}b$$

from which we deduce that

$$|Qb_s| \leq \sqrt{\frac{25}{3}}Kb^2$$

and hence

$$|Qb_s| = O(\exp(2\sqrt{K}s)).$$

Finally,

$$|a_{ss}| \leq aK$$

shows that

$$|a_s| = O(\exp(\sqrt{K}s)).$$

This concludes the proof. □

Now we begin proving the conserved inequalities listed above.

Lemma 2.5.2. *Let $(M_k, g(t))$, $t \in [0,T]$, $k \geq 1$, be a Ricci flow with bounded curvature. Then the inequality*

$$Q \leq 1$$

is preserved by the Ricci flow.

Proof. Define the quantity $\tilde{Q} = 1 - Q$. From the evolution equation (2.5.1) of Q we see

$$\partial_t \tilde{Q} = \tilde{Q}_{ss} + 3\frac{b_s}{b}\tilde{Q}_s + \tilde{Q}\left(-\frac{4}{b^2}Q(1+Q)\right).$$

As $Q \geq 0$ everywhere, the coefficient $-\frac{4}{b^2}Q(1+Q)$ is non-positive. Furthermore, by Lemma 2.5.1 we have $|\tilde{Q}| = o(\exp(s^2))$. Therefore we may apply the maximum principle of Theorem 2.3.2 to deduce that $\tilde{Q} \geq 0$ on $M_k \times [0,T]$. The desired result thus follows. □

Lemma 2.5.3. *Let $(M_k, g(t))$, $t \in [0,T]$, $k = 1,2$, be a Ricci flow with bounded curvature. Then the inequality*

$$x \leq 0$$

is preserved by the Ricci flow.

Proof. The evolution equation of x, as derived in the Appendix A, is

$$\begin{aligned} \partial_t x &= L[x] - \frac{2}{b^2}\left(2Q^2 + y^2\right)x - \frac{2}{b^2}\left(Q^2+2\right)y^2 \\ &\leq L[x] - \frac{2}{b^2}\left(2Q^2+y^2\right)x \end{aligned} \tag{2.5.7}$$

Note that $|x| = o(\exp(s^2))$ by Lemma 2.5.1. Therefore applying the maximum principle of Theorem 2.3.2 yields the desired result. □

Remark 2.5.4. Note that $x = 2 - k$ at $s = 0$ by the boundary conditions (2.1.3). Therefore the result can only hold for $k = 1, 2$.

Lemma 2.5.5. *Let $(M_k, g(t))$, $t \in [0,T]$, $k \geq 1$, be a Ricci flow with bounded curvature. Then the inequality*

$$y \leq 0$$

is preserved by the Ricci flow.

Proof. Let $K > 0$ such that

$$\sup_{M_k \times [0,T]} |Rm_{g(t)}|_{g(t)} < K.$$

Since y is an odd quantity, we consider the quantity $Qy = Qb_s - Q^2$ instead. Its evolution equation is

$$\partial_t Qy = L[Qy] - 2\frac{Qy}{b^2}\left(2(Q^2+x) + Qy + y^2\right). \tag{2.5.8}$$

Note that

$$\begin{aligned}
-\frac{2}{b^2}\left(2(Q^2+x)+Qy+y^2\right) &= -\frac{2}{b^2}\left(4Q^2-4+b^2M_1+Qb_s+b_s^2\right) \\
&\leq \frac{8}{b^2}+2K+2\frac{|Q||b_s|}{b^2} \\
&\leq \frac{8}{b^2}+2K+\frac{Q^2}{b^2}+\frac{b_s^2}{b^2},
\end{aligned}$$

where M_1 is one of the curvature components listed in section 2.2. By Lemma 2.3.3 we see that for some $C>0$

$$-\frac{2}{b^2}\left(2(Q^2+x)+Qy+y^2\right) \leq CK \text{ on } M_k\times[0,T]$$

Furthermore $|Qy| = o(\exp(s^2))$ by Lemma 2.5.1. Now the result follows from applying the maximum principle of Theorem 2.3.2. □

Lemma 2.5.6. *Let $(M_k, g(t))$, $t\in[0,T]$, $k\geq 1$, be a Ricci flow with bounded curvature. If the initial metric $g(0)$ satisfies $a_s, b_s \geq 0$, then $a_s, b_s\geq 0$ for all times $t\in[0,T]$.*

Proof. The evolution equations (2.5.3) and (2.5.4) of a_s and Qb_s can be written as a system of weakly coupled parabolic equations

$$\partial_t a_s = L[a_s] - \left(2\left(\frac{b_s}{b}\right)^2+6\frac{Q^2}{b^2}\right)a_s + 8\frac{Q^2}{b^2}(Qb_s) \tag{2.5.9}$$

$$\partial_t Qb_s = L[Qb_s] + 4\frac{Q^2}{b^2}a_s + \left(\frac{8-10Q^2}{b^2}-2\left(\frac{b_s}{b}\right)^2\right)(Qb_s), \tag{2.5.10}$$

By Lemma 2.3.3 and Lemma 2.3.4 the zeroth order coefficients of a_s and Qb_s are bounded. Lemma 2.5.1 shows that $|a_s|, |b_s| = o(\exp(s^2))$. Finally, note that the off-diagonal coefficients $8\frac{Q^3}{b^2}$ and $4\frac{Q^2}{b^2}$ are non-negative. Thus the desired result follows by the maximum principle for weakly coupled parabolic equations of Lemma 2.3.6. □

Lemma 2.5.7. *Let $(M_k, g(t))$, $t\in[0,T]$, $k\geq 1$, be a Ricci flow with bounded curvature satisfying $y\leq 0$, $Q\leq 1$ and $a_s, b_s\geq 0$. Then for $C\geq 2$ the inequality*

$$a_s \leq C$$

is preserved by the Ricci flow.

Proof. Define the quantity $A \coloneqq a_s - C$. Then from the evolution equation (2.5.9) of a_s it follows that

$$\partial_t A = L[A] - \left(2\left(\frac{b_s}{b}\right)^2+6\frac{Q^2}{b^2}\right)A + \frac{1}{b^2}\left(8Q^3b_s - CQ^2 - 2Cb_s^2\right).$$

Fix $C \geq 2$. Then

$$8Q^3 b_s - 6CQ^2 - 2Cb_s^2 \leq 8Q^4 - CQ^2 \leq (8-6C)Q^2 \leq 0,$$

where we used $Q \leq 1$ and $y = b_s - Q \leq 0$. As $|A| = o(\exp(s^2))$ by Lemma 2.5.1 it follows from the maximum principle of Theorem 2.7.5 that the inequality $A \leq 0$ is preserved by the Ricci flow. This proves the desired result. □

Lemma 2.5.8. *Let $(M_k, g(t))$, $t \in [0,T]$, $k \geq 1$, be a Ricci flow with bounded curvature satisfying $y \leq 0$, $b_s \geq 0$ and $Q \leq 1$. Then the inequality*

$$T_1 = a_s + 2Q^2 - 2 \geq 0$$

is preserved by the Ricci flow.

Proof. The evolution equation of T_1 is

$$\begin{aligned}\partial_t T_1 = L[T_1] &+ \frac{1}{b^2}\left[-4\left(1+Q^2\right)y^2 + 8Q\left(1-2Q^2\right)y + 16Q^2\left(1-Q^2\right)\right] \\ &+ T_1 \frac{2y}{b^2}\left(2Q - y\right),\end{aligned} \tag{2.5.11}$$

which can be derived from the evolution equations (2.5.3), (2.5.4) and (2.5.5) for a_s, Qb_s and Q^2 listed above. Inspecting the quadratic expression

$$-4\left(1+Q^2\right)y^2 + 8Q\left(1-2Q^2\right)y + 16Q^2\left(1-Q^2\right) \tag{2.5.12}$$

we see that when $y = 0$ it is equal to

$$16Q^2\left(1-Q^2\right) \geq 0$$

and when $y = -Q$ it is equal to

$$4Q^2\left(1-Q^2\right) \geq 0$$

As $y = b_s - Q \in [-Q, 0]$ by the assumptions $y \leq 0$, $b_s \geq 0$ and $Q \leq 1$, and furthermore the quadratic expression (2.5.12) is concave in y, we conclude that

$$\partial_t T_1 \geq L[T_1] + \frac{2y}{b^2}\left(2Q - y\right) T_1$$

Note that the zeroth order coefficient of T_1 is bounded by Lemma 2.3.3. Furthermore $|T_1| = o(\exp(s^2))$ by Lemma 2.5.1. Hence the result follows from applying the maximum principle of Theorem 2.3.2. □

Below we prove some further preserved conditions. These can be skipped on the first reading.

Lemma 2.5.9. *Let $(M_k, g(t))$, $t \in [0,T]$, $k = 1,2$, be a Ricci flow with bounded curvature satisfying $Q \leq 1$. Then the condition*

$$T_2 = Qy - x = -a_s + Qb_s + 2\left(1 - Q^2\right) \geq 0$$

is preserved by the Ricci flow.

Proof. Note that $T_2 = 2 - k$ when $s = 0$ by the boundary conditions (2.1.3). Therefore the result can only hold true for $k = 1, 2$. The evolution equations of T_2 is

$$\partial_t T_2 = L[T_2] + \frac{4}{b^2}\left(1 - Q^2\right)y^2 - 2\frac{T_2}{b^2}\left((b_s - 2Q)^2 + Q^2\right). \tag{2.5.13}$$

The coefficients are bounded by Lemma 2.3.3. Furthermore $|T_2| = o(\exp(s^2))$ by Lemma 2.5.1. Therefore applying the maximum principle of Theorem 2.3.2 yields the desired result. □

Lemma 2.5.10. *Let $(M_k, g(t))$, $t \in [0,T]$, $k \geq 1$, be a Ricci flow with bounded curvature satisfying $Q \leq 1$. Then the inequality*

$$T_3 = a_s - Qb_s - Q^2 + 1 \geq 0$$

is preserved by the Ricci flow.

Proof. The evolution equations of T_3 is

$$\partial_t T_3 = L[T_3] + \frac{2}{b^2}\left(1 - Q^2\right)y^2 - 2\frac{T_3}{b^2}\left((b_s + Q)^2 + 4Q^2\right) \tag{2.5.14}$$

Note that the coefficients are bounded by Lemma 2.3.3. Furthermore $|T_3| = o(\exp(s^2))$ by Lemma 2.5.1. Applying the maximum principle of Theorem 2.3.2 yields the desired result. □

Lemma 2.5.11. *Let $(M_k, g(t))$, $t \in [0,T]$, $k \geq 1$, be a Ricci flow with bounded curvature satisfying $y \leq 0$, $b_s \geq 0$ and $Q \leq 1$. Then the inequality*

$$\min(T_1, T_4) \geq 0$$

is preserved by the Ricci flow. Here

$$\begin{aligned} T_1 &= a_s + 2Q^2 - 2 \\ T_4 &= a_s - \frac{1}{2}Qb_s - \left(1 - Q^2\right). \end{aligned}$$

Proof. By Lemma 2.5.8 we already know that the inequality

$$T_1 = a_s - 2 + 2Q^2 \geq 0$$

is preserved. Thus we only need to show that $T_4 \geq 0$ is preserved whenever the Ricci flow satisfies $T_1 \geq 0$. The evolution equation of T_4 is

$$\partial_t T_4 = L[T_4] + \frac{1}{b^2}\left(b_s\left(5Q^3 - 2b_s\right) - 2T_4\left(4Q^2 - 2Qb_s + b_s^2\right)\right).$$

A computation shows

$$\frac{1}{2}Qb_s = T_1 - T_4 + 1 - Q^2.$$

By the assumption $y \leq 0$ we have

$$\frac{Q^2}{2} \geq \frac{1}{2}Qb_s$$

and hence it follows that

$$Q^2 \geq \frac{2}{3}\left(1 - T_4\right).$$

Therefore

$$5Q^3 - 2b_s \geq 5Q^3 - 2Q \geq Q\left(\frac{4}{3} - \frac{10}{3}T_4\right),$$

which implies that

$$\partial_t T_4 \geq L[T_4] - \frac{2T_4}{b^2}\left(4Q^2 - \frac{1}{3}Qb_s + b_s^2\right)$$

since $b_s \geq 0$. Note that the zeroth order coefficient of T_4 is bounded by Lemma 2.3.3. Furthermore $|T_4| = o(\exp(s^2))$ by Lemma 2.5.1. Applying the maximum principle of Theorem 2.3.2 yields the desired result. $\square$

2.6 Exclusion of shrinking solitons

In this section we rule out $U(2)$-invariant shrinking solitons on M_k, $k \geq 2$, within a large class of metrics. In particular, we show

Theorem 2.6.1 (No shrinker)**.** *On M_k, $k \geq 2$, there does not exists a complete $U(2)$-invariant shrinking Ricci soliton of bounded curvature satisfying the conditions*

1. $\sup_{p \in M_k} |b_s| < \infty$
2. $T_1 = a_s + 2Q^2 - 2 > 0$ *for* $s > 0$
3. $Q = \frac{a}{b} \leq 1$

This theorem is the key ingredient in section 2.10, where we show that certain Ricci flows on M_k, $k \geq 3$, develop Type II singularities in finite time.

Soliton equations. Recall that a shrinking Ricci soliton $(M, g(t))$ is a solution to the Ricci flow equation that up to diffeomorphism homothetically shrinks. Such a soliton solution may be written as

$$g(t) = \sigma^2(t)\Phi_t^* g(0),$$

where

$$\sigma(t) = \sqrt{1 - 2\rho t}$$

for some $\rho > 0$ and Φ_t is a family of diffeomorphisms. The reader may consult [Top06] for more details. Hence for a $U(2)$-invariant shrinking Ricci soliton $(M_k, g(t))$, $k \geq 1$, the corresponding warping functions can be written as

$$a(s,t) = \sigma(t)a\left(\frac{s}{\sigma(t)}, 0\right) \tag{2.6.1}$$

$$b(s,t) = \sigma(t)b\left(\frac{s}{\sigma(t)}, 0\right). \tag{2.6.2}$$

The above formulae are with respect to the radial coordinate s, which is equivalent to fixing a gauge. For this reason the family of diffeomorphisms Φ_t does not appear explicitly. Differentiating with respect to t at time 0 yields

$$\begin{aligned}\partial_t|_{t=0}a(s,t) &= a_s(s,0)\left(\frac{\partial s}{\partial t} + \rho s\right) - \rho a(s,0)\\ &= a_s(s,0)f_s - \rho a(s,0),\end{aligned}$$

where $f : M_k \to \mathbb{R}$ is the potential function satisfying

$$f_{ss} = \rho + \frac{a_{ss}}{a} + 2\frac{b_{ss}}{b}$$

and we used the expression (2.2.15) for $\frac{\partial s}{\partial t}$ derived in section 2.2. Similarly we obtain

$$\partial_t|_{t=0}b(s,t) = b_s(s,0)f_s(s) - \rho b(s,0).$$

Substituting the expressions $\partial_t a$ and $\partial_t b$ from the Ricci flow equations (2.2.11) and (2.2.12), respectively, we see that the soliton equations for the warping functions a and b at time $t = 0$ read

$$f_{ss} = \frac{a_{ss}}{a} + 2\frac{b_{ss}}{b} + \rho \tag{2.6.3}$$

$$a_{ss} = 2\frac{a^3}{b^4} - 2\frac{a_s b_s}{b} + a_s f_s - \rho a \tag{2.6.4}$$

$$b_{ss} = \frac{4}{b} - 2\frac{a^2}{b^3} - \frac{a_s b_s}{a} - \frac{b_s^2}{b} + b_s f_s - \rho b \tag{2.6.5}$$

In a slight abuse of notation we will denote a and b as functions of s only when we are considering Ricci solitons. In that case a and b should be interpreted as the initial data $a(s,0)$ and $b(s,0)$ at time zero that leads to a Ricci soliton solution, via the correspondence (2.6.1) and (2.6.2).

Remark 2.6.2. The above shows that all $U(2)$-invariant Ricci solitons on M_k are automatically gradient Ricci solitons with potential function f.

Evolution of x, y and Q on soliton background. Since x, y and Q are *scale-invariant* quantities, their evolution on a Ricci soliton background can be expressed as follows:

$$\begin{aligned} x(s,t) &= x\left(\frac{s}{\sigma(t)}, 0\right) \\ y(s,t) &= y\left(\frac{s}{\sigma(t)}, 0\right) \\ Q(s,t) &= Q\left(\frac{s}{\sigma(t)}, 0\right) \end{aligned}$$

Differentiating, we therefore obtain

$$\begin{aligned} \partial_t|_{t=0} x(s,t) &= x_s(s,0) f_s(s) \\ \partial_t|_{t=0} y(s,t) &= y_s(s,0) f_s(s) \\ \partial_t|_{t=0} Q(s,t) &= Q_s(s,0) f_s(s). \end{aligned}$$

With help of the evolution equations (2.5.7), (2.13.1) and (2.5.1) for x, y and Q, this yields the following ordinary differential equations for x, y and Q at time zero on a soliton background

$$0 = x_{ss} + \left(2\frac{b_s}{b} - \frac{a_s}{a} - f_s\right) x_s - \frac{1}{b^2}\left(2Q^2\left(2x + y^2\right) + 2y^2\left(2 + x\right)\right) \tag{2.6.6}$$

$$0 = y_{ss} + \left(\frac{a_s}{a} - f_s\right) y_s - \frac{y}{a^2}\left((x+2)^2 + Q^2\left(2x + y^2\right)\right) \tag{2.6.7}$$

$$0 = Q_{ss} + \left(3\frac{b_s}{b} - f_s\right) Q_s + \frac{4}{b^2} Q\left(1 - Q^2\right). \tag{2.6.8}$$

Alternatively these equations can be derived from the soliton equations (2.6.3)-(2.6.5). In a slight abuse of notation we will often denote x, y and Q as functions of s only when we are considering Ricci solitons.

Exclusion of shrinking solitons. By [CZ10] we know that the potential function of a non-compact complete shrinking Ricci soliton grows quadratically with the distance to some fixed point. In our setting this translates into the following lemma:

Lemma 2.6.3. *Let (M_k, g), $k \geq 1$, be a complete non-compact shrinking Ricci soliton of bounded curvature. Then*

$$\begin{aligned} f &\sim \frac{\rho}{2} s^2 \\ f_s &\sim \rho s \end{aligned}$$

as $s \to \infty$.

Proof. See Theorem 1.1, equation (2.3) and equation (2.8) of [CZ10]. □

This allows us to prove the following lemma:

Lemma 2.6.4. *Let (M_k, g), $k \geq 1$, be a complete non-compact shrinking Ricci soliton of bounded curvature with $Q \leq 1$ on M_k. Then $Q_s \geq 0$ on M_k.*

Proof. First notice that for a complete shrinking Ricci soliton with $Q \leq 1$, the strong maximum principle applied to the evolution equation (2.5.1) of Q forces

$$Q < 1 \text{ for } s \geq 0,$$

as otherwise we would have $Q = 1$ everywhere, which cannot be the case. Similarly,

$$Q > 0$$

unless we are at the origin $s = 0$. By equation (2.6.8) we have

$$Q_{ss} = \left(f_s - 3\frac{b_s}{b}\right) Q_s - \frac{4}{b^2} Q\left(1 - Q^2\right). \tag{2.6.9}$$

We now argue by contradiction. Assume there exists an $s_* > 0$ such that $Q_s(s_*) < 0$. Then $Q_s(s) < 0$ for all $s > s_*$, because at any extremum of Q we have $Q_s = 0$ and

$$Q_{ss} = -\frac{4}{b^2} Q\left(1 - Q^2\right) < 0.$$

Lemma 2.3.3 shows that $\frac{b_s}{b}$ is bounded and from Lemma 2.6.3 it follows that

$$f_s \to \infty \text{ as } s \to \infty.$$

Therefore eventually

$$f_s - 3\frac{b_s}{b} > 0$$

from which it follows by equation (2.6.9) that

$$Q_{ss} < 0$$

for sufficiently large s. This, however, contradicts that $Q > 0$ unless $s = 0$. □

In the lemma below we bound the term

$$G := (x+2)^2 + Q^2(2x + y^2),$$

which appears in the evolution equation (2.6.7) of y, away from zero.

Lemma 2.6.5. *Whenever $Q_s \geq 0$ and $Q, T_1 > 0$ we have $G > 0$.*

Proof. We have

$$\frac{Q_s}{Q} = \frac{a_s}{a} - \frac{b_s}{b} = \frac{x}{a} - \frac{y}{b} + \frac{2}{a} - \frac{2a}{b^2}.$$

For $Q_s \geq 0$ it follows that

$$x - Qy \geq 2\left(Q^2 - 1\right).$$

Recall the quantity

$$T_1 = a_s + 2Q^2 - 2$$

defined in section 2.5. Then

$$\begin{aligned}
G &\geq (x+2)^2 + Q^2\left(2\left(Qy + 2\left(Q^2-1\right)\right) + y^2\right) \\
&= x^2 + 4x + 4 + 2Q^3y + 4Q^4 - 4Q^2 + Q^2y^2 \\
&= \left(a_s + Q^2 - 2\right)^2 + 4\left(a_s + Q^2 - 2\right) + 4 + 3Q^4 - 4Q^2 + Q^2(y+Q)^2 \\
&= a_s^2 + 2Q^2a_s + 4\left(Q^4 - Q^2\right) + Q^2(y+Q)^2 \\
&= a_s^2 + 2Q^2T_1 + Q^2(y+Q)^2 \\
&= a_s^2 + Q^2b_s^2 + 2Q^2T_1 > 0
\end{aligned}$$

□

Now we prove the non-existence of shrinking solitons.

Proof of Theorem 2.6.1. We argue by contradiction. Assume such a shrinking Ricci soliton exists. Applying L'Hôpital's Rule to the evolution equation (2.2.12) of b shows that at $s = 0$

$$\begin{aligned}
\partial_t b\Big|_{s=0} &= 2b_{ss} - \frac{4}{b} \\
&= 2\left(y_s + \frac{k-2}{b}\right).
\end{aligned}$$

Clearly, every shrinking soliton satisfies

$$\partial_t b\Big|_{s=0} < 0.$$

The boundary conditions (2.1.3) of a and b at $s = 0$ imply that

$$y(0) = 0.$$

and thus we deduce from the above that

$$y_s(0) < 0,$$

as $k \geq 2$ by assumption. The ordinary differential equation (2.6.7) for y can be written as

$$y_{ss} = \left(f_s - \frac{a_s}{a}\right)y_s + \frac{y}{a^2}G. \tag{2.6.10}$$

Lemma 2.6.4 and Lemma 2.6.5 imply that

$$G > 0 \text{ for } s > 0,$$

which in turn shows that $y_s \leq 0$ everywhere, as at a negative local minimum of y we would have

$$y_{ss} = \frac{y}{a^2} G < 0.$$

The asymptotic properties of f listed in Lemma 2.6.3 and the bounds on $\frac{a_s}{a}$ proven in Lemma 2.3.4 show that eventually

$$f_s - \frac{a_s}{a} > 0$$

and hence from the equation (2.6.10) it follows that

$$y_{ss} < 0$$

for s sufficiently large. From this it follows that

$$\lim_{s \to \infty} y = \lim_{s \to \infty} (b_s - Q) = -\infty,$$

which contradicts our assumptions on b_s and Q. □

2.7 Curvature bound

The aim of this section is to prove that a Ricci flow $(M_k, g(t))$, $k \in \mathbb{N}$, $t \in [0, T)$, starting from an initial metric $g(0) \in \mathcal{I}$ — where $\mathcal{I}$ is a class of metrics to be discussed below — with $\sup_{p \in M_k} b(p, 0) < \infty$ satisfies the curvature bound

$$|Rm_{g(t)}|_{g(t)} \leq C_1 b^{-2} \text{ for } t \in (0, T),$$

where $C_1 > 0$ is some constant. This allows us to control the geometry via the warping function b, which will be crucial for constructing blow-up limits in the following parts of the chapter. Note that this bound was already derived in the compact case in [IKS17] and we will follow their strategy to prove it in our non-compact setting.

Recall the following definition (see also [ChI][Definition 8.23]):

Definition 2.7.1 (κ-non-collapsing)**.** Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow and $\kappa > 0$. We say that the Ricci flow is κ-non-collapsed at a point (p_0, t_0) in spacetime at scale ρ if the following two conditions hold for all $r \leq \rho$:

- (bounded normalized curvature) We have $|Rm(p, t)| \leq r^{-2}$ for every $(p, t) \in B_{g(t_0)}(p_0, r) \times [t_0 - r^2, t_0]$. In particular we assume $[t_0 - r^2, t_0] \subset [0, T)$.
- (non collapsed volume) At time t_0 the ball $B_{g(t_0)}(p_0, r)$ has volume at least κr^4.

We now define the class of metrics $\mathcal{I}$.

Definition 2.7.2. For $K > 0$ let $\mathcal{I}_K$ be the set of all complete *bounded curvature* metrics of the form (2.2.3) on M_k, $k \geq 1$, with *positive injectivity radius* that satisfy the following scale-invariant inequalities:

$$Q \leq 1 \tag{2.7.1}$$

$$a_s, b_s \geq 0 \tag{2.7.2}$$

$$y \leq 0 \tag{2.7.3}$$

$$\sup a_s < K \tag{2.7.4}$$

$$\sup |bb_{ss}| < K \tag{2.7.5}$$

Denote by $\mathcal{I}$ the set of metrics g such that for sufficiently large $K > 0$ we have $g \in \mathcal{I}_K$.

Note that for any $k \in \mathbb{N}$ the set $\mathcal{I}$ of metrics on M_k is non-empty, as for example the metric on M_k defined by

$$\begin{aligned} a(s) &= Q = \tanh(ks), \quad k \in \mathbb{N} \\ b(s) &= 1 \end{aligned}$$

is contained in $\mathcal{I}$. In Lemma 2.7.9 below we show that if $g(0) \in \mathcal{I}_{K_0}$ for some $K_0 > 0$ then there exists a $K > K_0$ such that $g(t) \in \mathcal{I}_K$ for $t \in [0, T)$. Note that conditions (2.7.1)-(2.7.5) are scale-invariant, and therefore pass to blow-up limits.

An adaptation of [ChI, Theorem 8.26] to our setting yields the following result:

Theorem 2.7.3 (No local collapsing)**.** *Let $g(t)$, $t \in [0, T)$, $T < \infty$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$. Then there exists a $\kappa > 0$ depending on T,* $\mathrm{inj}(g(0))$ *and* $\sup_{M\times[0,T/2]} Ric_{g(t)}$ *such that $g(t)$ is κ-non-collapsed at every $(p, t) \in M \times (\frac{T}{2}, T)$ at every scale $\rho < \sqrt{T/2}$.*

Remark 2.7.4. Recall that if a Ricci flow $g(t)$ is κ-non-collapsed at scale ρ, then the parabolically dilated Ricci flow $\alpha^2 g(\alpha^{-2}t)$ is κ-non-collapsed at scale $\alpha\rho$. As the κ-non-collapsedness property is preserved under Cheeger-Gromov limits, a blow-up limit of a Ricci flow $(M_k, g(t))$, $[0, T_{sing})$ is κ-non-collapsed at all scales.

Having set up the necessary terminology, we may now state the main theorem of this section:

Theorem 2.7.5 (Curvature bound)**.** *Let $(M_k, g(t))$, $t \in [0, T)$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ (see Definition 2.7.2) with*

$$\sup_{p\in M_k} b(p, 0) < \infty.$$

Then there exists a constant $C_1 > 0$ such that

$$|Rm_{g(t)}|_{g(t)}(p) \leq C_1 b(p, t)^{-2}$$

for $(p, t) \in M_k \times (0, T)$.

A useful variant of Theorem 2.7.5 is:

Corollary 2.7.6. *Let $(M_k, g(t))$ with $g(t) \in \mathcal{I}$ (see Definition 2.7.2) for $t \in (-\infty, 0]$ be an ancient Ricci flow solution which is κ-non-collapsed at all scales. Then there exists a constant $C_1 > 0$ such that*

$$|Rm_{g(t)}|_{g(t)}(p) \leq C_1 b(p,t)^{-2}$$

for $(p,t) \in M_k \times (-\infty, 0]$.

Remark 2.7.7. Corollary 2.7.6 follows immediately from Theorem 2.7.5 for ancient κ-non-collapsed Ricci flows that arise as blow up limits of Ricci flows $(M_k, g(t))$, $t \in [0, T_{sing})$, $g(0) \in \mathcal{I}$, as the curvature bound is scale-invariant. Nevertheless, we give a proof of the general case.

Let us now prove the assertions made above. We begin with the following lemma:

Lemma 2.7.8. *Let $K_0 > 0$ and assume that $(M_k, g(t))$, $k \geq 1$, $t \in [0, T)$, is a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}_{K_0}$. Then there exists a constant $K \geq 0$, depending only on the initial metric $g(0)$, such that*

$$|bb_{ss}| \leq K \tag{2.7.6}$$

on $M_k \times [0, T)$.

Proof. We follow the proof strategy of [IKS17, Lemma 7]. Consider the quantities

$$\begin{aligned} H_- &= bb_{ss} + a_s^2 - b_s^2 - C \\ H_+ &= bb_{ss} - a_s^2 - b_s^2 + C, \end{aligned}$$

where $C > 0$ is a constant to be determined later. The goal is to show that the inequalities $H_+ \geq 0$ and $H_- \leq 0$ are preserved by the Ricci flow for sufficiently large $C > 0$. The quantities $H_\pm$ satisfy the evolution equations

$$\begin{aligned} \partial_t H_\pm = {} & [H_\pm]_{ss} + \left(\frac{a_s}{a} - 2\frac{b_s}{b}\right)[H_\pm]_s + H_\pm \left(-\frac{2a_s^2}{a^2} - \frac{4a^2}{b^4} - \frac{4b_s^2}{b^2}\right) \\ & \pm C\left(\frac{2a_s^2}{a^2} + \frac{4a^2}{b^4} + \frac{4b_s^2}{b^2}\right) \\ & \pm 2a_{ss}^2 + a_{ss}\left(-\frac{2ba_sb_s}{a^2} \mp \frac{8a_sb_s}{b} \pm \frac{4a_s^2}{a} + \frac{4a}{b^2}\right) \\ & + \frac{2ba_s^3b_s}{a^3} - \frac{32aa_sb_s}{b^3} \mp \frac{16a^3a_sb_s}{b^5} + \frac{4a_s^2}{b^2} \pm \frac{8a^2a_s^2}{b^4} \\ & \mp \frac{2a_s^4}{a^2} + \frac{32a^2b_s^2}{b^4} - \frac{16b_s^2}{b^2}. \end{aligned}$$

In the Appendix A we carry out the derivation of the evolution equation. We now show that $H_- \leq 0$ is preserved. Using Young's inequality to bound the terms involving a_{ss} and then disregarding non-positive terms not involving C, we obtain

$$\begin{aligned}
\partial_t H_- \leq{}& [H_-]_{ss} + \left(\frac{a_s}{a} - 2\frac{b_s}{b}\right)[H_-]_s + H_-\left(-\frac{2a_s^2}{a^2} - \frac{4a^2}{b^4} - \frac{4b_s^2}{b^2}\right) \\
& - C\left(\frac{2a_s^2}{a^2} + \frac{4a^2}{b^4} + \frac{4b_s^2}{b^2}\right) \\
& + \frac{1}{2}\left(\left(\frac{2ba_sb_s}{a^2}\right)^2 + \left(\frac{8a_sb_s}{b}\right)^2 + \left(\frac{4a_s^2}{a}\right)^2 + \left(\frac{4a}{b^2}\right)^2\right) \\
& + \frac{2ba_s^3b_s}{a^3} + \frac{16a^3a_sb_s}{b^5} + \frac{4a_s^2}{b^2} + \frac{2a_s^4}{a^2} + \frac{32a^2b_s^2}{b^4}
\end{aligned}$$

Recall that on $M_k \times [0,T)$ we have $y = b_s - Q \leq 0$, $Q \leq 1$, $a_s, b_s \geq 0$ and $a_s \leq C'$ for some $C' > 0$ by Lemma 2.5.5, Lemma 2.5.2, Lemma 2.5.6 and Lemma 2.5.7, respectively. Therefore we obtain the following bounds away from the non-principal orbit S_o^2:

$$\begin{aligned}
\left(\frac{2ba_sb_s}{a^2}\right)^2 &= \left(\frac{2a_sb_s}{aQ}\right)^2 \leq \frac{4a_s^2}{a^2} \\
\left(\frac{8a_sb_s}{b}\right)^2 &= \left(8\frac{a_s}{a}Qb_s\right)^2 \leq 64\frac{a_s^2}{a^2} \\
\left(\frac{4a_s^2}{a}\right)^2 &= 16C'^2\frac{a_s^2}{a^2} \\
\frac{2ba_s^3b_s}{a^3} &= \frac{2a_s^3}{a^2}\frac{b_s}{Q} \leq 2C'\frac{a_s^2}{a^2} \\
\frac{16a^3a_sb_s}{b^5} &= 16Q^3\frac{a_sb_s}{b^2} \leq 8Q^3\left(\frac{a_s^2}{b^2} + \frac{b_s^2}{b^2}\right) \leq 8\left(\frac{a_s^2}{a^2} + \frac{b_s^2}{b^2}\right) \\
\frac{4a_s^2}{b^2} &\leq 4\frac{a_s^2}{a^2} \\
\frac{2a_s^4}{a^2} &\leq 2C'^2\frac{a_s^2}{a^2} \\
\frac{32a^2b_s^2}{b^4} &= 32Q^2\frac{b_s^2}{b^2} \leq 32\frac{b_s^2}{b^2}
\end{aligned}$$

Hence for a sufficiently large $C > 0$ it follows that

$$\partial_t H_- \leq [H_-]_{ss} + \left(\frac{a_s}{a} - 2\frac{b_s}{b}\right)[H_-]_s + H_-\left(-\frac{2a_s^2}{a^2} - \frac{4a^2}{b^4} - \frac{4b_s^2}{b^2}\right)$$

away from the non-principal orbit S_o^2. Switching to coordinates (s,t) we see that for $s > 0$

$$\partial_t\Big|_s H_- \leq [H_-]_{ss} + \left(\frac{a_s}{a} - 2\frac{b_s}{b}\right)[H_-]_s + H_-\left(-\frac{2a_s^2}{a^2} - \frac{4a^2}{b^4} - \frac{4b_s^2}{b^2} - \frac{\partial s}{\partial t}\right). \tag{2.7.7}$$

On the non-principal orbit S^2_o, or equivalently when $s = 0$, we have

$$H_- = bb_{ss} + k^2 - C \leq bQ_s + k^2 - C \leq k + k^2 - C,$$

where we used that $y = b_s - Q \leq 0$ with equality at $s = 0$. Choosing $C > k^2 + k$ we have $H_- < 0$ on $\{s = 0\} \times [0, T)$. Hence for every $T' \in [0, T)$ there exists a $s_0 > 0$ such that

$$H_-(s, t) \leq 0 \text{ on } [0, s_0] \times [0, T'],$$

as $H_-(s, t)$ is a smooth function on $\mathbb{R}_{\geq 0} \times [0, T']$. Furthermore note that

$$|H_-| \leq |Rm_{g(t)}|_{g(t)} b^2 + C'^2 + 1 + C,$$

where we used the expression for the curvature component R_{0202} derived in section 2.2. This shows that for each time $t < T'$ the function $H_-(s, t)$ grows subexponentially. Note that by Lemma 2.3.3 and Lemma 2.3.4 the coefficient

$$\frac{a_s}{a} - 2\frac{b_s}{b}$$

is bounded on $[s_0, \infty) \times [0, T']$. Similarly, we see from the bound (2.3.7) on $|\frac{\partial s}{\partial t}|$ presented in the proof of the maximum principle of Theorem 2.3.2, Case 1, that the coefficient

$$-\frac{2a_s^2}{a^2} - \frac{4a^2}{b^4} - \frac{4b_s^2}{b^2} - \frac{\partial s}{\partial t}$$

grows at most linearly on every times slice of $[s_0, \infty) \times [0, T']$. Therefore, applying the weak maximum principle to the evolution equation (2.7.7) of H_- on the parabolic neighborhood $[s_0, \infty) \times [0, T']$, we deduce that

$$H_- \leq 0 \text{ on } M_k \times [0, T'].$$

As $T' \in [0, T)$ was arbitrary it follows that $H_- \leq 0$ is preserved by the Ricci flow.

We repeat the same process to prove that $H_+ \geq 0$ is preserved. Applying Young's inequality to bound terms involving a_{ss} and then disregarding non-negative terms not involving C, we see that

$$\begin{aligned}
\partial_t H_+ \geq\ & [H_+]_{ss} + \left(\frac{a_s}{a} - 2\frac{b_s}{b}\right)[H_+]_s + H_+\left(-\frac{2a_s^2}{a^2} - \frac{4a^2}{b^4} - \frac{4b_s^2}{b^2}\right) \\
& + C\left(\frac{2a_s^2}{a^2} + \frac{4a^2}{b^4} + \frac{4b_s^2}{b^2}\right) \\
& - \frac{1}{2}\left(\left(\frac{2ba_sb_s}{a^2}\right)^2 + \left(\frac{8a_sb_s}{b}\right)^2 + \left(\frac{4a_s^2}{a}\right)^2 + \left(\frac{4a}{b^2}\right)^2\right) \\
& - \frac{32aa_sb_s}{b^3} - \frac{16a^3a_sb_s}{b^5} - \frac{2a_s^4}{a^2} - \frac{16b_s^2}{b^2}.
\end{aligned}$$

Bounding the zeroth order terms via Young's inequality as above, we see that for $C > 0$ sufficiently large

$$\partial_t H_+ \geq [H_+]_{ss} + \left(\frac{a_s}{a} - 2\frac{b_s}{b}\right)[H_+]_s + H_+\left(-\frac{2a_s^2}{a^2} - \frac{4a^2}{b^4} - \frac{4b_s^2}{b^2}\right)$$

away from the non-principal orbit S^2_o. On the non-principal orbit S^2_o we have

$$H_+ = bb_{ss} - k^2 + C \geq -k^2 + C,$$

where we used that $b_s \geq 0$ with equality at $s = 0$ to deduce that $b_{ss} \geq 0$ at $s = 0$. From here the above proof that $H_- \leq 0$ is preserved carries over and we may conclude that $H_+ \geq 0$ is preserved as well. Recalling the bounds on a_s and b_s, the desired result now follows. □

Now we can prove that $\mathcal{I}$ (see Definition 2.7.2) is preserved by Ricci flow:

Lemma 2.7.9. *Let $K_0 > 0$. Then there exists a $K \geq K_0$ such that the following holds: Let $(M_k, g(t))$, $k \geq 1$, $t \in [0, T)$, be a Ricci flow solution starting from an initial metric $g(0) \in \mathcal{I}_{K_0}$. Then $g(t) \in \mathcal{I}_K$ for every $t \in [0, T)$.*

Proof. By Lemma 2.5.2, Lemma 2.5.6, Lemma 2.5.5, Lemma 2.5.7 we see that for $K > 2$ the conditions (2.7.1) - (2.7.4) are preserved. By Lemma 2.7.8 we see that there exists a $K \geq K_0$ such that inequality (2.7.5) holds for $t \in [0, T)$.

Now we only need to prove that for every time $t \in [0, T)$ the metric $g(t)$ has bounded curvature and positive injectivity radius. As the curvature of $g(0)$ is bounded by the assumption that $g(0) \in \mathcal{I}_K$, it follows by Shi's Theorem [Shi89] that for every time $T' \in [0, T)$ the Ricci flow $g(t)$ has bounded curvature on the time interval $[0, T']$. As $\mathrm{inj}_{g(0)} > 0$ it follows that the metric $g(0)$ is non-collapsed. By standard volume distortion estimates it follows that for each $t \in [0, T/2]$ the metric $g(t)$ is non-collapsed, and hence $\mathrm{inj}_{g(t)} > 0$. By Theorem 2.7.3 there exists a $\kappa > 0$ and $\rho > 0$ such that for each $t \in [0, T)$ the metric $g(t)$ is κ-non-collapsed at scale $\rho < \sqrt{T/2}$. This shows that $\mathrm{inj}_{g(t)} > 0$ for all $t \in [T/2, T)$. □

Before proving Theorem 2.7.5, we need to prove the following two lemmas in preparation:

Lemma 2.7.10. *Let $(M_k, g(t))$, $k \geq 1$, $t \in [0, T)$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$. Then there exists a constant $C_0 \geq 0$, depending only on the initial metric $g(0)$, such that*

$$|\partial_t b^2| \leq C_0 \tag{2.7.8}$$

Proof. By Lemma 2.7.9 there exists a $K > 0$ such that $g(t) \in \mathcal{I}_K$ for $t \in [0, T)$. From the evolution equation (2.2.12) of b and Definition 2.7.2 of $\mathcal{I}_K$ it follows that

$$\begin{aligned} |\partial_t b^2| &= \left|2bb_{ss} - 8 + 4Q^2 + 2\frac{a_s b_s}{Q} + 2b_s^2\right| \\ &\leq 2K + 8 + 4 + 2K + 2 \\ &= 4K + 14 \end{aligned}$$

This concludes the proof. □

Lemma 2.7.11. *Let* $(M_k, g(t))$, $k \geq 1$, $t \in [0, T)$, *be a Ricci flow starting from an initial metric* $g(0) \in \mathcal{I}$. *Then*

$$\sup_{p \in M_k} b(p,t) \leq \sup_{p \in M_k} b(p,0)$$

for all $t \in [0, T)$.

Proof. From the evolution equation (2.2.12) of b and expression (2.2.4) for the Laplacian with respect to the background metric $g(t)$ it follows

$$\begin{aligned}\partial_t b^2 &= \Delta_{g(t)} b^2 - 8 + 4Q^2 - 4b_s^2 \\ &\leq \Delta_{g(t)} b^2 - 4.\end{aligned}$$

Applying the maximum principle [ChII, Theorem 12.14] yields the desired result. □

We now proceed to proving Theorem 2.7.5.

Proof of Theorem 2.7.5. We argue by contradiction. Assume there exists a sequence of points (p_i, t_i) in spacetime and constants $D_i \to \infty$ as $i \to \infty$ such that

$$|Rm_{g(t_i)}|_{g(t_i)}(p_i) = D_i b(p_i, t_i)^{-2} := K_i$$

and

$$|Rm_{g(t)}|_{g(t)} \leq D_i b^{-2} \text{ on } M_k \times [0, t_i].$$

By the assumption that $g(0) \in \mathcal{I}$ the initial metric $g(0)$ has bounded curvature. Hence by Shi's theorem [Shi89] we have that for every $T' \in [0, T)$ the metric $g(t)$ has bounded curvature on $M_k \times [0, T']$. As by Lemma 2.7.11 the warping function b is uniformly bounded on $M_k \times [0, T)$, we thus see that $D_i \to \infty$ forces $K_i \to \infty$ and therefore $t_i \to T$.

Consider the rescaled Ricci flows

$$g_i(t) = K_i g\left(t_i + K_i^{-1} t\right), \quad t \in [-K_i \Delta t_i, 0],$$

where $\Delta t_i > 0$ is to be determined below. As $K_i \to \infty$ we see that $g_i(t)$ are blow-ups rather than blow-downs, which is important for the following reason: By Theorem 2.7.3 there exists a $\kappa > 0$ such that $g(t)$ is κ-non-collapsed at every scale $p \leq \sqrt{T/2}$ at every spacetime point $(p, t) \in M_k \times [T/2, T)$. As $K_i \to \infty$ we see that $g_i(t)$ are κ-non-collapsed at scales tending to infinity as $i \to \infty$.

By Lemma 2.7.9 there exists a $K > 0$ such that $g(t) \in \mathcal{I}_K$ for all $t \in [0, T)$. Furthermore, by Lemma 2.7.10 there exists a C_0 such that $|\partial_t b^2| \leq C_0$ on $M_k \times [0, T)$. Recall the Definition 2.2.1 of $C_g(p, r)$. Set

$$\Delta t_i = \min\left(\frac{t_i}{2}, \frac{b^2(p_i, t_i)}{8C_0}\right)$$

and consider the parabolic neighborhoods

$$\Omega_i = C_{g(t_i)}\left(p_i, \frac{b(p_i,t_i)}{2}\right) \times [t_i - \Delta t_i, t_i].$$

As $g(t) \in \mathcal{I}_K$ for $t \in [0,T)$ we have that $y = b_s - Q \leq 0$, $Q \leq 1$ and $b_s \geq 0$ everywhere on $M_k \times [0,T)$. Therefore

$$b(p,t_i) \geq \frac{b(p_i,t_i)}{2} \text{ on } \Omega_i \cap \{t = t_i\}$$

By Lemma 2.7.10

$$b^2(p,t_i) - b^2(p,t) \leq C_0(t_i - t)$$

for all $(p,t) \in \Omega_i$ from which it follows that

$$\frac{1}{4}b(p_i,t_i)^2 - b(p,t)^2 \leq b^2(p,t_i) - b^2(p,t) \leq C_0(t_i - t) \leq C_0\Delta t \leq \frac{1}{8}b(p_i,t_i)^2.$$

Thus we deduce that

$$b^2(p,t) \geq \frac{1}{8}b^2(p_i,t_i) \text{ on } \Omega_i. \tag{2.7.9}$$

It follows that for $(p,t) \in \Omega_i$

$$\begin{aligned} |Rm_{g(t)}|_{g(t)}(p) &\leq D_i b(p,t)^{-2} \\ &\leq 8D_i b(p_i,t_i)^{-2} \\ &= 8K_i \end{aligned}$$

and hence the curvatures of the rescaled metrics $g_i(t)$ satisfy

$$|Rm_{g_i(t)}|_{g_i(t)} \leq 8$$

on the parabolic neighborhoods Ω_i'

$$\Omega_i' := C_{g_i(0)}\left(p_i, \sqrt{K_i}\frac{b(p_i,t_i)}{2}\right) \times [-K_i\Delta t_i, 0].$$

Note that

$$K_i\Delta t_i \to \infty \text{ as } i \to \infty$$

and

$$\sqrt{K_i}\frac{b(p_i,t_i)}{2} \geq \frac{\sqrt{D_i}}{2} \to \infty \text{ as } i \to \infty.$$

Hence $\left(C_{g_i(t)}(p_i, \sqrt{D_i}/2), g_i(t), p_i\right)$, $t \in [-K_i\Delta t_i, 0]$, subsequentially converges, in the Cheeger-Gromov sense, to an ancient pointed Ricci flow $(M_\infty, g_\infty(t), p_\infty)$, $t \in (-\infty, 0]$.

Claim 1: The Ricci flow $(M_\infty, g_\infty(t), p_\infty)$, $t \in (-\infty, 0]$, splits as $(\mathbb{R}^2 \times N, g_{eucl} + g_N(t))$, $t \in (-\infty, 0]$, where g_{eucl} is the flat euclidean metric, and $(N, g_N(t))$ is a non-compact ancient Ricci flow.

Proof of Claim: Denote by a_i and b_i the warping functions of the rescaled metrics $g_i(t)$. Then by (2.7.9) we see that

$$b_i(p,t) \geq \sqrt{\frac{D_i}{8}} \text{ on } \Omega_i' \tag{2.7.10}$$

As $D_i \to \infty$, the warping functions b_i tend to infinity uniformly. As b_i describes the size of the base S^2 in the Hopf fibration,intuitively one can see that this claim is true. Nevertheless, we provide a formal proof below:

As $g(t) \in \mathcal{I}_K$ for $t \in [0,T)$ we have

$$\left| -\frac{b_{ss}}{b} \right| \leq \frac{K}{b^2} \text{ on } M_k \times [0,T).$$

Inspecting the curvature components listed in section 2.2, we see that all the curvature components of $g_i(t)$, apart from R_{0101}, tend to zero on Ω_i'. Hence the curvature operator of $g_\infty(t)$ is of rank 1. Furthermore, as $g(0)$ has bounded curvature by the assumption that $g(0) \in \mathcal{I}$ we see that the scalar curvature $R_{g(t)}$ is pointwise bounded below by $\inf_{p \in M_k} R_{g(0)}(p) > -\infty$. Hence the blow-up limit $g_\infty(t)$ has non-negative scalar curvature, which in turn implies that the curvature operator is non-negative. By [Ham86, 8.3. Theorem & p. 178] we conclude that $(M_\infty, g_\infty(t))$ splits as a product $(\mathbb{R}^2 \times N, g_{eucl} + g_N(t))$. Note also that N is diffeomorphic to the leafs of the distribution spanned by e_0 and e_1, as these are the only planes with non-flat sectional curvature. Recalling that $e_0 = \frac{\partial}{\partial s}$ we see that the integral curves of e_0 are non-compact and therefore N is non-compact as well. ■

As $(M_\infty, g_\infty(t))$ is κ-non-collapsed at all scales, the above claim implies that $(N, g_N(t))$ is a 2d κ-solution. However, by Hamilton's work a two dimensional κ-solution is either the shrinking round sphere S^2 or its $\mathbb{Z}_2$ quotient [CLN06, §1 of Chapter 9]. Since N is non-compact we have arrived at a contradiction. Therefore the desired result follows. □

Proof of Corollary 2.7.6. The proof is the same as for Theorem 2.7.5. Since the Ricci flow is assumed to be κ-non-collapsed at all scales, we may also take blow-down limits and do not need to assume that b is uniformly bounded. Furthermore, since ancient Ricci flows have non-negative scalar curvature, Claim 1 of the proof of Theorem 2.7.5 also carries over. □

2.8 Compactness properties

In this section we prove some compactness properties of $U(2)$-invariant cohomogeneity one Ricci flows. For general Ricci flows the compactness properties are well-known [ChI, Chapter 3]. Therefore the main technical difficulty is to show that the $U(2)$-symmetry passes to the limit.

The main theorem of this section is Theorem 2.8.1 which roughly states the following compactness property: Let $(U_i, g_i(t), p_i)$, $[-\Delta t, 0]$, be a sequence of $U(2)$-invariant cohomogeneity one manifolds in the class $\mathcal{I}$ of metrics. Here the U_i are open manifolds and assumed to compactly contain the sets $C_{g_i(0)}(p_i, r)$ (see Definition 2.2.1) for some fixed $r > 0$. This condition can be understood as requiring U_i to have 'radial diameter' of at least r. Furthermore the metrics $g_i(t)$ are normalized such that $b = 1$ at the points $(p_i, 0)$ in spacetime. We show that if the flows $g_i(t)$ are κ-non-collapsed and of uniformly bounded curvature, then $(U_i, g_i(t), p_i)$, $[-\Delta t, 0]$, subsequentially converges to a limiting $U(2)$-invariant Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$. Moreover, if we correctly pick/normalize the coordinate ξ, the warping functions $u_i(\xi, t)$, $a_i(\xi, t)$ and $b_i(\xi, t)$ of the metrics $g_i(t)$ converge on compact parabolic sets in C^∞ to the warping functions $u_\infty(\xi, t)$, $u_\infty(\xi, t)$ and $b_\infty(\xi, t)$ of $g_\infty(t)$. This in essence shows that when taking limits of $U(2)$-invariant Ricci flows, we may work with the warping functions only, without having to concern ourselves with the underlying manifold.

Theorem 2.8.1 has two important applications: Firstly, it implies the corresponding compactness result for complete Ricci flows. In particular, a sequence of uniformly bounded and non-collapsed $U(2)$-invariant cohomogeneity one Ricci flows $(M_k, g_i(t), p_i)$, $t \in [-t_i, 0]$, $t_i \to \infty$, normalized such that $b(p_i, 0) = 1$, subsequentially converges, in the Cheeger-Gromov sense, to a limiting Ricci flow $(M_\infty, g_\infty(t), p_\infty)$, $t \in [-\infty, 0]$, that is also $U(2)$-invariant and cohomogeneity one. Secondly, we prove a variant of Theorem 2.8.1 in Proposition 2.8.3, where we specialize to the case in which the 'radial diameter' of the U_i is equal to $\frac{1}{2}$. This will allow us to alter one assumption of Theorem 2.8.1 and yield a very useful tool for proving certain scale-invariant inequalities via a contradiction/compactness argument, as introduced in the outline of section 9 in section 2.1 of this chapter.

Below we state the main results of this section. For this recall Definition 2.2.1 of $C_g(p, r)$, $C_g^+(p, r)$ and Σ_p.

Theorem 2.8.1 (Local compactness)**.** *Let $k \in \mathbb{N}$ and $\kappa, \rho, K, r, \Delta t > 0$. Assume that*

$$(U_i, g_i(t), p_i), \quad t \in [-\Delta t, 0],$$

is a sequence of pointed cohomogeneity one $U(2)$-invariant Ricci flows satisfying the following properties:

1. *U_i is an open $U(2)$-invariant manifold with principal orbit $S^3/\mathbb{Z}_k$.*
2. *For $t \in [-\Delta t, 0]$ we have $g_i(t) \in \mathcal{I}$ (see Definition 2.7.2). Denote by u_i, a_i and b_i the warping functions of $g_i(t)$.*
3. *The closed sets $\overline{C_{g_i(0)}(p_i, r)} \subset U_i$ are compact.*
4. *$b_i(p_i, 0) = 1$.*
5. *The Ricci flow $(U_i, g_i(t))$ is κ-non-collapsed at $(p_i, 0)$ at scale $\min(\rho, r, \sqrt{\Delta t})$.*
6. *$|Rm_{g_i(t)}|_{g_i(t)} \leq K$ in $U_i \times [-\Delta t, 0]$.*

Then $(C_{g_i(0)}(p_i, r), g_i(t), p_i)$, $t \in [-\Delta t, 0]$, subsequentially converges, in the Cheeger-Gromov sense, to a pointed Ricci flow

$$(\mathcal{C}_\infty, g_\infty(t), p_\infty), \quad t \in [-\Delta t, 0],$$

satisfying the following properties:

(a) *$\mathcal{C}_\infty$ is a cohomogeneity one $U(2)$-invariant manifold such that either*

  (i) *All orbits are principal: In this case $\mathcal{C}_\infty$ is diffeomorphic to the cylinder $\mathbb{R} \times S^3/\mathbb{Z}_k$ and we equip $\mathcal{C}_\infty$ with a radial coordinate $\xi : \mathcal{C}_\infty \to \mathbb{R}$ defined by $\xi(p) = d_{g_\infty(0)}(p, \Sigma_{p_\infty})$.*

  (ii) *There is exactly one non-principal orbit: In this case $\mathcal{C}_\infty$ is diffeomorphic to M_k and we equip $\mathcal{C}_\infty$ with the radial coordinate $\xi : \mathcal{C}_\infty \to \mathbb{R}$ defined by $\xi(p) = d_{g_\infty(0)}(p, S^2_o)$.*

(b) *There exist warping functions*

$$u_\infty, a_\infty, b_\infty : C_{g_\infty(0)}(p_\infty, r) \times [-\Delta t, 0] \to \mathbb{R}_{\geq 0}$$

such that the metric $g_\infty(t)$, $t \in [-\Delta t, 0]$, is of the form (2.2.2) and in the class $\mathcal{I}$

(c) *Choosing the coordinate ξ on $(U_i, g_i(t), p_i)$ corresponding to whether we are in case (i) or (ii) above, the warping functions $u_i(\xi, t)$, $a_i(\xi, t)$ and $b_i(\xi, t)$ converge on compact sets to $u_\infty(\xi, t)$, $a_\infty(\xi, t)$ and $b_\infty(\xi, t)$.*

(d) *For every $r' < r$ the closed set $\overline{C_{g_\infty(0)}(p_\infty, r')} \subset \mathcal{C}_\infty$ is compact.*

From Theorem 2.8.1 the following corollary follows immediately:

Corollary 2.8.2 (Compactness of complete Ricci flows)**.** *Let $k \in \mathbb{N}$, $\kappa, K > 0$ and $r_i, t_i, \rho_i \to \infty$ as $i \to \infty$. Assume that $(\mathcal{M}_i, g_i(t), p_i)$, $t \in [-t_i, 0]$, is a sequence of pointed $U(2)$-invariant cohomogeneity one Ricci flows satisfying:*

1. *For $t \in [-t_i, 0]$ we have $g_i(t) \in \mathcal{I}$. Denote by u_i, a_i and b_i the warping functions of $g_i(t)$.*
2. *$\overline{C_{g_i(0)}(p_i, r_i)} \subset \mathcal{M}_i$ is compact.*
3. *$b(p_i, 0) = 1$*
4. *$g_i(t)$ is κ-non-collapsed at scale ρ_i*
5. *$|Rm_{g_i(t)}|_{g_i(t)} \leq K$ on $\mathcal{M}_i \times [-t_i, 0]$*

Then $(\mathcal{M}_i, g_i(t), p_i)$, $t \in [-t_i, 0]$, subsequentially converges, in the Cheeger-Gromov sense, to a pointed complete ancient Ricci flow $(\mathcal{M}_\infty, g_\infty(t), p_\infty)$, $t \in (-\infty, 0]$, with bounded curvature satisfying properties (a) - (d) of Theorem 2.8.1, when taking $\mathcal{C}_\infty = \mathcal{M}_\infty$ and $r = \infty$.

Proof. This follows from Theorem 2.8.1 by a diagonal argument. $\square$

The following proposition is a variant of Theorem 2.8.1 in the case we take $r = \frac{1}{2}$.

Proposition 2.8.3. *Let $k \in \mathbb{N}$, $\kappa, \rho, C_1 > 0$, $r = \frac{1}{2}$ and $\Delta t \in (0, \frac{1}{48C_1}]$. Assume*

$$(U_i, g_i(t), p_i), \quad t \in [-\Delta t, 0],$$

is a sequence of pointed $U(2)$-invariant cohomogeneity one Ricci flows satisfying conditions (1)-(5) of Theorem 2.8.3. If, instead of condition (6) of Theorem 2.8.1, we require

(6') $|Rm_{g_i(t)}|_{g_i(t)} \leq \frac{C_1}{b^2}$ on $U_i \times [-\Delta t, 0]$

then

$$\left(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i\right), \quad t \in [-\Delta t, 0],$$

subsequentially converges, in the Cheeger-Gromov sense, to a pointed Ricci flow

$$(\mathcal{C}_\infty, g_\infty(t), p_\infty), \quad t \in [-\Delta t, 0],$$

satisfying the same properties (a)-(d) listed in Theorem 2.8.3.

Proof of Proposition 2.8.3. For brevity we write $\Omega_i = C_{g_i(0)}\left(p_i, \frac{1}{2}\right) \times [-\Delta t, 0]$. Let a_i and b_i denote the warping functions of $g_i(t)$. As $g_i(t) \in \mathcal{I}$ we have

$$0 \leq (b_i)_s \leq Q_i \leq 1 \text{ in } \Omega_i,$$

where $Q_i = \frac{a_i}{b_i}$ and thus

$$b_i(p, 0) \geq \frac{1}{2} \text{ for } p \in C_{g_i(0)}\left(p_i, \frac{1}{2}\right)$$

as $b_i(p_i, 0) = 1$ by assumption. By the Ricci flow equation we have

$$\begin{aligned} \partial_t b_i^2 &= -2b^2 \left(R_{0202} + R_{1212} + R_{2323}\right) \\ &\leq 6C_1 \text{ on } \Omega_i. \end{aligned}$$

This implies

$$b_i(p, t) \geq \frac{1}{\sqrt{8}} \text{ for } (p, t) \in \Omega_i,$$

as $\Delta t \leq \frac{1}{48C_1}$ by assumption. This yields the uniform curvature bound

$$|Rm(g_i)|_{g_i} \leq 8C_1 \text{ on } \Omega_i.$$

The result now follows from Theorem 2.8.1. $\square$

The main proof idea of Theorem 2.8.1 is to construct a set of four Killing vector fields $\overline{X}_j$, $j = 1, 2, 3, 4$, generated by the $U(2)$-action on each $(U_i, g_i(t))$, and show that these Killing vector fields pass to the limit $(\mathcal{C}_\infty, g_\infty(t))$. This allows us to reconstruct the $U(2)$-action on $\mathcal{C}_\infty$, proving the desired result. The main difficulty, however, is to show that the orbits corresponding to the flows of the Killing vector fields do not degenerate in the limit and thereby ensure that the full $U(2)$ symmetry group is preserved. For this we will rely on Lemma 2.8.4 below, where we prove that κ-non-collapsedness implies a lower positive bound on Q away from a non-principal orbit.

Lemma 2.8.4. *Let $k \in \mathbb{N}$, $r_0 \in (0, 1]$ and $\kappa, C_1, c > 0$. Assume that (M, g) is a $U(2)$-invariant cohomogeneity one manifold with principal orbit $S^3/\mathbb{Z}_k$ equipped with a metric $g \in \mathcal{I}$. Take $p \in M$. If*

1. *The set $\overline{C_g^+(p, b(p)r_0)}$ (See Definition 2.7.2) is compactly contained in M*
2. *$|Rm_g|_g \leq \frac{C_1}{b^2}$ on $\overline{C_g^+(p, b(p)r_0)}$*
3. *g is κ-non-collapsed at scale $cb(p)$: If for $r \leq cb(p)$ the ball $B_g(p, r)$ is compactly contained in M and $|Rm_g|_g < r^{-2}$ on $B_g(p, r)$ then $vol(B_g(p, r)) \geq \kappa r^4$*

then there exists an $\epsilon > 0$ depending on k, κ, C_1, c and r_0 for which the following holds: If for $q \in C_g^+(p, b(p)r_0)$ the set $C_g(q, b(p)\frac{r_0}{4})$ is compactly contained in $C_g^+(p, b(p)r_0)$ then $Q(q) \geq \epsilon$.

Proof. By rescaling we may assume without loss of generality $b(p) = 1$ and that the metric g is κ-non-collapsed at scale $c > 0$. The latter follows from the fact that if g is κ-non-collapsed at scale ρ then $\alpha^2 g$ is κ-non-collapsed at scale $\alpha\rho$. Fix a $q \in C_g^+(p, b(p)r_0)$ such that the assumptions of the lemma hold. Take $U := C_g(q, b(p)\frac{r_0}{4})$. Note that U is a union of orbits of the $U(2)$-action. Recall that non-principal orbits are non-generic and characterized by $a = 0$. As $a_s \geq 0$ we see that all the orbits of U are principal and therefore diffeomorphic to $S^3/\mathbb{Z}_k$. Because $0 \leq b_s \leq Q \leq 1$ for metrics in $\mathcal{I}$ we see that

$$1 \leq b \leq 2 \text{ in } C_g^+(p, r_0)$$

and hence

$$|Rm_g|_g \leq C_1 \text{ in } C_g^+(p, r_0)$$

by assumption (2). From the expression

$$M_2 = \frac{1}{b^2}(a_s - Qb_s)$$

for the curvature component R_{0231} derived in section 2.2 and the fact that

$$Q_s = \frac{1}{b}(a_s - Qb_s)$$

we deduce that

$$|Q_s| \leq C_1 \text{ in } C_g^+(p, r_0).$$

Thus for $r \leq r_1 := \min\left(\frac{r_0}{4}, \frac{Q(q)}{C_1}\right)$ we have

$$Q \leq 2Q(q) \ \text{ on } C_g(q, r_1).$$

Claim 1: For $r \leq r_2 := \min\left(\frac{1}{100}, r_1\right)$ we have

$$vol(B_g(q,r)) \leq Cr^3 Q(q)$$

for some constant $C > 0$ depending on k only.

Proof of Claim: Let $q' \in C_g(q,r)$, $r < r_2$. Then $\Sigma_{q'}$ is isometric to $S^3/\mathbb{Z}_k$ equipped with a squashed Berger metric. In particular, if we denote by $\iota : \Sigma_{q'} \to M$ the inclusion, then

$$\iota^* g = a(q')^2 \omega \otimes \omega + b(q')^2 \pi^*(g_{FS}),$$

where g_{FS} is the Fubini-Study metric on S^2 normalized to have curvature equal to $\frac{1}{4}$ and $\pi : S^3/\mathbb{Z}_k \to S^2$ is the Hopf fibration. Note that

$$\Sigma_{q'} \cap B_g(q,r) \subseteq \Sigma_{q'} \cap B_g(q',r) \subseteq \pi^{-1}(B_{g_{FS}}(\pi(q'),r)) \subseteq \Sigma_{q'}.$$

Furthermore, as the Hopf fibers of $\Sigma_{q'} \cong S^3/\mathbb{Z}_k$ have length $\frac{2\pi}{k}a(q')$, we see that

$$vol(\pi^{-1}(B_{g_{FS}}(\pi(q'),r))) = \frac{2\pi}{k} a(q') vol(B_{g_{FS}}(\pi(q'),r)) \leq Ca(q')r^2,$$

for some constant $C > 0$ depending on k only. Since $Q = \frac{a}{b}$, $Q \leq 2Q(q)$ and $b \in [1,2]$ in $C_g(q,r_1)$ it follows that

$$vol(\Sigma_{q'} \cap B_g(q,r)) \leq 4CQ(q)r^2.$$

Integrating this inequality proves the claim.

■

As $|Rm_g|_g \leq C_1$ on $C_g^+(p,r_0)$, the ball $B_g(q, \frac{r_0}{4})$ is compactly contained in M, and g is κ-non-collapsed at scale c, we see that for $r \leq r_3 := \min\left(\frac{1}{\sqrt{C_2}}, c, \frac{r_0}{4}\right)$

$$vol(B_g(q,r)) \geq \kappa r^4.$$

Setting $r_4 := \min(r_2, r_3)$ we therefore obtain

$$Cr_4^3 Q(q) \geq vol(B_g(q,r_4)) \geq \kappa r_4^4.$$

Rearranging this inequality proves the lemma. □

Before proving the compactness theorems listed above, we construct a set of four Killing vector fields on a general $U(2)$-invariant cohomogeneity one manifold M with principal orbit $S^3/\mathbb{Z}_k$. By passing to the universal cover we may assume without loss of generality that $k = 1$. Pick the basis

$$X_0 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \qquad X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for the Lie algebra of $U(2)$. Then X_i, $i = 1, 2, 3, 4$, satisfy the commutation relations

$$[X_1, X_2] = 2X_3 \qquad [X_2, X_3] = 2X_1 \qquad [X_3, X_1] = 2X_2.$$

and

$$[X_0, X_i] = 0 \ \text{ for } \ i = 1, 2, 3.$$

Extend X_i, $i = 1, 2, 3, 4$, to left-invariant vector fields on $U(2)$. Note that the integral curves generated by X_i, $i = 1, 2, 3, 4$, have period 2π. The $U(2)$-action generates four corresponding Killing vector fields $\overline{X}_i$, $i = 1, 2, 3, 4$, on M_k by taking

$$\overline{X}_i(p) = \frac{d}{dt}\Big|_{t=0} \exp(tX_i) \cdot p, \quad p \in M_k, \ i = 1, 2, 3, 4.$$

We now prove the following:

Lemma 2.8.5. *For $i = 1, 2, 3, 4$ we have*

$$|\overline{X}_i|_g \leq \max(a, b).$$

Proof. By the form (2.2.3) of the metric we see that $|\overline{X}_0|_g = a$. Hence we only need to prove the result for $i = 1, 2, 3$. First note that the vector fields $\overline{X}_i$, $i = 1, 2, 3$, are orthogonal to $\frac{\partial}{\partial s}$ and therefore parallel to the orbits of the $U(2)$ action on M_k. Hence it suffices to study the metric g restricted to these directions. Here we see that

$$a^2 \omega \otimes \omega + b^2 \pi^*(g_{FS}) \leq \max(a, b)^2 g_{S^3},$$

where g_{S^3} is the round metric on S^3 with sectional curvatures equal to 1. Thus it suffices to show that

$$|\overline{X}_i|_{g_{S^3}} \leq 1.$$

If we identify S^3 with $SU(2)$, the vectors $\overline{X}_i$ correspond to right-invariant vector fields on $SU(2)$. Moreover, one can check that these vector fields are orthonormal with respect to the metric g_{S^3}. Hence the desired result follows. $\square$

Remark 2.8.6. In fact one can show that $\min(a, b) \leq |\overline{X}_i|_g \leq \max(a, b)$. Recalling that the isometry generated by the Killing vector field $\overline{X}_i$ descends to a rotation of the base S^2 in the Hopf fibration $\pi : S^3 \to S^2$, one can see that the upper bound is attained on $\pi^{-1}(\{\text{Equator of } S^2\})$ and the lower bound is attained on $\pi^{-1}(\{N, S\})$, where N, S denote the north and south pole with respect to the rotation induced by $\overline{X}_i$.

Now we proceed to proving the main theorem of this section:

Proof of Theorem 2.8.1. As $g_i(t)$ is κ-non-collapsed at $(p_i, 0)$ at scale $\min(\rho, r, \sqrt{\Delta t})$ it follows from [ChI, Lemma 6.54] that there exists a uniform $\delta > 0$ such that

$$\mathrm{inj}_{g_i(0)}(p_i) > \delta.$$

By assumption $g_i(t)$ has bounded curvature on the parabolic neighborhood

$$\Omega_i := C_{g_i(0)}(p_i, r) \times [-\Delta t, 0].$$

By an adaptation of [ChI, Theorem 3.16] we therefore deduce that after passing to a subsequence

$$\left(C_{g_i(0)}(p_i, r), g_i(t), p_i\right), \quad t \in [-\Delta t, 0],$$

converges, in the Cheeger-Gromov sense, to a pointed Ricci flow

$$(\mathcal{C}_\infty, g_\infty(t), p_\infty), \quad t \in [-\Delta t, 0],$$

where $\mathcal{C}_\infty$ is an open manifold.

Claim 1: $(\mathcal{C}_\infty, g_\infty(t))$, $t \in [-\Delta t, 0]$, is $U(2)$-invariant.

Proof of Claim: Recall the construction of the Killing vector fields $\overline{X}_j$, $j = 1, 2, 3, 4$ for a general $U(2)$-invariant manifold M explained above. Let $\overline{X}_{ij}$, $i \in \mathbb{N}$, $j = 1, 2, 3, 4$, denote the corresponding Killing vector fields on the manifolds $C_{g_i(0)}(p_i, r)$. Recall that $g_i(t) \in \mathcal{I}$ implies that $0 \leq b_s \leq Q \leq 1$. Therefore $b \leq r + 1$ on $C_{g_i(0)}(p_i, r)$. Note that from the evolution equation (2.2.12) of b it follows that

$$\left|\frac{\partial_t b}{b}\right| \leq c|Rm_{g_i(t)}|_{g_i(t)},$$

where $c > 0$ is some universal constant. As $|Rm_{g_i(t)}|_{g_i(t)} \leq C_1$ on Ω_i by assumption, we see that there exists a $C > 0$, depending on r and C_1 only, such that $b \leq C$ on Ω_i. From Lemma 2.8.5 it hence follows that for $i \in \mathbb{N}$, $j = 1, 2, 3, 4$,

$$|\overline{X}_{ij}|_{g_i(t)} \leq C \text{ on } \Omega_i.$$

Recall that in general a Killing vector field X^a on a manifold satisfies the relation

$$\nabla_a \nabla_b X^c = -R^c_{abd} X^d.$$

Therefore we see that the Killing vector fields $\overline{X}_{ij}$ are uniformly bounded in $C^2(\Omega_i)$, and converge to C^1 Killing vector fields $\overline{X}_{\infty,j}$, $j = 1, 2, 3, 4$, on $(\mathcal{C}_\infty, g_\infty(t))$ after passing to a subsequence. However, since the group of isometries of a smooth manifold is a smooth Lie group, the vector fields $\overline{X}_{\infty,j}$, $j = 1, 2, 3, 4$ are in fact smooth. As the Killing vector fields

$\overline{X}_{ij}$, $i \in \mathbb{N}$, $j = 1,2,3,4$, are complete, so are $\overline{X}_{\infty,j}$, $j = 1,2,3,4$. Integrating the Killing vector fields $\overline{X}_{\infty,j}$, $j = 1,2,3,4$, then yields the desired $U(2)$-action on $(\mathcal{C}_\infty, g_\infty(t))$. ■

It remains to be shown that this action is faithful by proving that the Killing vector fields are non-zero at times $t \in [-\Delta t, 0]$.

Claim 2: The $U(2)$-action on $(\mathcal{C}_\infty, g_\infty(t))$, $t \in [-\Delta t, 0]$ is faithful.

Proof of Claim: Take $r_1 > 0$ such that $C_{g_\infty(t)}(p_\infty, r_1)$ is compactly contained in $\mathcal{C}_\infty$ for all $t \in [-\Delta t, 0]$. This is possible by standard distance distortion estimates and the fact that $\mathcal{C}_\infty \times [-\Delta t, 0]$ has bounded curvature. Furthermore, since $C_{g_\infty(t)}(p_\infty, r_1) \times [-\Delta t, 0]$ is compactly contained in $\mathcal{C}_\infty \times [-\Delta t, 0]$, there exist constants $\rho', \kappa' > 0$ such that for each $t \in [-\Delta t, 0]$ the manifold $(C_{g_\infty(t)}(p_\infty, r_1), g_\infty(t))$ is κ'-non-collapsed at scale less or equal to ρ'. Since $(C_{g_i(t)}(p_i, r), g_i(t), p_i)$ converges in the Cheeger-Gromov sense to $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$ we see that eventually $(C_{g_i(t)}(p_i, r_1), g_i(t))$ is $\kappa'/2$-non-collapsed at scales less or equal to $\rho'/2$.

Fix $t' \in [-\Delta t, 0]$ and choose points $q_i \in \Sigma^+_{p_i}$ (see Definition 2.2.1) and $q_\infty \in \mathcal{C}_\infty$ with $d_{g_i(t')}(q_i, \Sigma^+_{p_i}) = \frac{1}{2} r_1$ and $q_i \to q_\infty$. Checking the conditions of Lemma 2.8.4, we see that there exists an $\epsilon > 0$, independent of i, such that

$$Q(q_i, t') \geq \epsilon.$$

As $g \in \mathcal{I}$ and therefore $0 \leq b_s \leq Q \leq 1$, we see that

$$1 \leq b(q_i, t') \leq \frac{3}{2}$$

Therefore the geometry of the orbit $\Sigma_{q_i} \cong S^3/\mathbb{Z}_k$ is controlled — the curvature and diameter are uniformly bounded from above, and its volume and Hopf fiber lengths are uniformly bounded away from zero. Hence the norms of the Killing vector fields $\overline{X}_{ij}$, $j = 1,2,3,4$, at the points (q_i, t') in spacetime are uniformly bounded away from zero, proving that on the limiting space $(\mathcal{C}_\infty, g_\infty(t'))$ the Killing vector fields $\overline{X}_{\infty,j}$, $j = 1,2,3,4$, are non-zero. As $t' \in [\Delta t, 0]$ was arbitrary, the desired result follows. ■

By the slice theorem we we see that either (i) all orbits of $\mathcal{C}_\infty$ are principal and diffeomorphic to $S^3/\mathbb{Z}_k$ or (ii) there exists exactly one non-principal orbit, which is diffeomorphic to S^2 and as usual we denote by S^2_o. Below it will become clear why $\mathcal{C}_\infty$ cannot possess two non-principal orbits. In case (i) $\mathcal{C}_\infty$ is diffeomorphic to the manifold $\mathbb{R} \times S^3/\mathbb{Z}_k$ and in case (ii) it is diffeomorphic to M_k. In both cases there is a dense open set of the form $\mathbb{R} \times S^3/\mathbb{Z}_k \subset \mathcal{C}_\infty$.

We now show that the metrics $g_\infty(t)$ can be expressed in the form (2.2.3). Denote the warping functions of the metrics $g_i(t)$ by a_i and b_i. In case (i) we define the radial coordinates

$$\xi_i(p) = \pm d_{g_i(0)}(p, \Sigma_{p_i}),$$

and

$$\xi_\infty(p) = \pm d_{g_\infty(0)}(p, \Sigma_{p_\infty})$$

on $C_{g_i(0)}(p_i, r)$ and $\mathcal{C}_\infty$, respectively. We choose the sign of $\xi_i(p)$ depending on which side of the hypersurface Σ_{p_i} the point p lies, and in such a way that $\partial_{\xi_i} a_i, \partial_{\xi_i} b_i \geq 0$. The sign of $\xi_\infty(p)$ is chosen such that $\xi_i \to \xi_\infty$ as $i \to \infty$. In case (ii) we may assume without loss of generality that for all $i \in \mathbb{N}$ the open manifolds $C_{g_i(0)}(p_i, r)$ contain a point o_i such that the orbit Σ_{o_i} is non-principal and $o_i \to o_\infty \in \mathcal{C}_\infty$ as $i \to \infty$. Then define radial coordinates

$$\xi_i(p) = d_{g_i(0)}(p, \Sigma_{o_i})$$

and

$$\xi_\infty(p) = d_{g_\infty(0)}(p, \Sigma_{o_\infty})$$

on $C_{g_i(0)}(p_i, r)$ and $\mathcal{C}_\infty$. Note that the coordinates ξ_i and ξ_∞ are smooth away from a non-principal orbit and furthermore that $\xi_i \to \xi_\infty$ in C^∞ away from a non-principal orbit. Hence we obtain the one-forms $d\xi_i$ and $d\xi_\infty$ away from a non-principal orbit, which are orthogonal to all orbits of $C_{g_i(0)}(p_i, r)$ and $\mathcal{C}_\infty$, respectively. For brevity we drop the subscript and write ξ for the coordinates ξ_i or ξ_∞.

Since the metric $g_\infty(t)$ is $U(2)$-invariant, as shown above, there exists warping functions $u_\infty, a_\infty, b_\infty : \mathcal{C}_\infty \times [-\Delta t, 0] \to \mathbb{R}_{\geq 0}$ such that the metric can be expressed as

$$g_\infty(t) = u_\infty^2(\xi, t) d\xi^2 + a_\infty^2(\xi, t)\omega \otimes \omega + b_\infty^2(\xi, t)\pi^*(g_{FS}),$$

where at time 0 we have

$$u = 1 \ \text{ on } \ \mathcal{C}_\infty.$$

As

$$a_\infty(p, t) = |\overline{X}_{\infty,0}|_{g_\infty(t)}(p)$$

and $\overline{X}_{i,o} \to \overline{X}_{\infty,0}$ as $i \to \infty$ by above, we see that away from a non-principal orbit $a_i \to a_\infty$ smoothly. Similarly, one can show with help of the remaining Killing vector fields $\overline{X}_{i,j}$, $j = 1, 2, 3$, that away from a non-principal orbit $b_i \to b_\infty$ smoothly.

Hence away from a non-principal orbit,$a_i(\xi, t), b_i(\xi, t) \to a_\infty(\xi, t), b_\infty(\xi, t)$ in C^∞ as $i \to \infty$. Furthermore, from the curvature bounds on Ω_i and the boundary conditions on a_i, b_i, a_∞, b_∞ at a non-principal orbit (see section 2.2 for the smoothness conditions on the warping functions at the non-principal orbit), one can show that in fact $a_i(\xi, t), b_i(\xi, t) \to a_\infty(\xi, t), b_\infty(\xi, t)$ smoothly everywhere. Hence the metric $g_\infty(t)$, $t \in [-\Delta t, 0]$, is in the class $\mathcal{I}$. As $a_s \geq 0$ for metrics in $\mathcal{I}$ we see that $\mathcal{C}_\infty$ can possess at most one non-principal orbit. Finally, we note that by [ChI, Theorem 3.16] the closed set $\overline{C_{g_\infty(0)}(p_\infty, r')} \subset \mathcal{C}_\infty$ is compact for every $r' < r$. $\square$

2.9 Ancient Ricci flows Part I

In this section we prove some properties of ancient Ricci flows $g(t) \in \mathcal{I}$, $-\infty < t \leq 0$, that are *non-collapsed at all scales.* This yields important geometric information on the blow-up limits of singular Ricci flows, which we exploit and refine in later chapters. The main goal is to prove the following theorem:

Theorem 2.9.1. *Let $\kappa > 0$ and $(M_k, g(t))$, $k \geq 2$, $t \in (-\infty, 0]$, be an ancient Ricci flow, which satisfies the following properties:*

(i) κ-non-collapsed at all scales

(ii) $g(t) \in \mathcal{I}$ for $t \in (-\infty, 0]$.

Then if $k = 2$ the following inequalities hold:

$$\begin{aligned} T_1 &= a_s + 2Q^2 - 2 \geq 0 \\ T_2 &= Qy - x \geq 0 \\ T_3 &= a_s - Qb_s - Q^2 + 1 \geq 0 \end{aligned}$$

If $k > 2$ we only have $T_1 \geq 0$ and $T_3 \geq 0$. For all $k \geq 2$ we have $T_1(p,t) = 0$ if, and only if, $k = 2$ and $p \in S^2_o$.

Furthermore we show

Theorem 2.9.2. *Let $\kappa > 0$ and $(M_2, g(t))$, $t \in (-\infty, 0]$, be an ancient Ricci flow, which satisfies the following properties:*

(i) κ-non-collapsed at all scales

(ii) $g(t) \in \mathcal{I}$ for $t \in (-\infty, 0]$

(iii) Kähler with respect to J_1, or equivalently $y = 0$ everywhere

Then $(M_2, g(t))$ is stationary and homothetic to the Eguchi-Hanson space.

Proof strategy. In both of these theorems we are given an ancient Ricci flow $(M, g(t))$, $t \leq 0$, and want to show that a scale invariant quantity T satisfies

$$T \geq 0 \text{ on } M \times (-\infty, 0].$$

We prove such statements by a contradiction/compactness argument. First we assume that

$$\iota := \inf_{M_k \times (-\infty, 0]} T < 0$$

and take a sequence of points (p_i, t_i) in spacetime such that

$$T(p_i, t_i) \to \iota \text{ as } i \to \infty.$$

Then we consider the rescaled Ricci flows

$$g_i(t) = \frac{1}{b^2(p_i, t_i)} g\left(t_i + b^2(p_i, t_i)t\right), \quad t \in [-\Delta t, 0],$$

where $\Delta t > 0$ is chosen such that the conditions of Proposition 2.8.3 are met. Then $(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i)$ subsequentially converges to a Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$, $t \in [-\Delta t, 0]$. By construction

$$T(p_\infty, 0) = \inf_{\mathcal{C}_\infty \times [-\Delta t, 0]} T = \iota < 0.$$

However, if the evolution equation of T precludes the possibility of a negative infimum being attained, we have arrived at a contradiction and proven the desired result.

Proof of main theorems of this section. Before proving Theorem 2.9.1 we need to state a technical lemma in preparation:

Lemma 2.9.3. *Let $(M_k, g(t))$, $k \in \mathbb{N}$, $t \leq 0$, be an ancient Ricci flow satisfying the conditions of Theorem 2.9.1. Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever at a point (p, t) in spacetime one of the following inequalities holds*

(i) $T_1(p, t) \leq -\epsilon$ and $k \geq 2$

(ii) $T_2(p, t) \leq -\epsilon$ and $k \leq 2$

(iii) $T_3(p, t) \leq -\epsilon$ and $k \in \mathbb{N}$

(iv) $|x(p, t)| \geq \epsilon$ and $k = 2$

then $s(p, t) \geq \delta b(p, t)$.

Proof. Recall that by Corollary 2.7.6 there exists a $C_1 > 0$ such that $|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{0}$ on $M_k \times (-\infty, 0]$. We first prove (i). We fix $\epsilon > 0$ and argue by contradiction. Assume there exists a sequence of points (p_i, t_i) in spacetime such that

$$T_1(p_i, t_i) \leq -\epsilon$$

and

$$\frac{s(p_i, t_i)}{b(p_i, t_i)} \to 0. \tag{2.9.1}$$

Define the rescaled metrics

$$g_i = \frac{1}{b^2(p_i, t_i)} g\left(t_i + tb^2(p_i, t_i)\right), \ t \in [-\Delta t, 0].$$

For sufficiently small $\Delta t > 0$ the conditions of Proposition 2.8.3 are satisfied and hence $\left(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i\right)$ subsequentially converges to a Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$. By (2.9.1) one sees that p_∞ lies on the non-principal orbit S_o^2 of $\mathcal{C}_\infty$. By construction we have $T_1(p_\infty, 0) \leq -\epsilon$ as T_1 is a scale invariant quantity. This however contradicts the fact that $T_1 = a_s + 2\left(Q^2 - 1\right) = k - 2 \geq 0$ on S_o^2.

Note that $T_2 = 2 - k$, $T_3 = k + 1$ and $x = k - 2$ on S_o^2. Therefore by the same argument applied to T_2, T_3 and x the desired result holds true. □

Next we prove Theorem 2.9.1.

Proof of Theorem 2.9.1. Recall that by Corollary 2.7.6 there exists a $C_1 > 0$ such that $|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{0}$ on $M_k \times (-\infty, 0]$. We first show that $T_1 \geq 0$ in $M_k \times (-\infty, 0]$. We argue by contradiction. Assume that

$$\iota := \inf_{M_k \times (-\infty, 0]} T_1 < 0.$$

As $g(t) \in \mathcal{I}$ we know that $\iota > -\infty$. Take a sequence of points (p_i, t_i) in spacetime such that

$$T_1(p_i, t_i) \to \iota \ \text{ as } \ i \to \infty.$$

From Lemma 2.9.3 it follows that for sufficiently large i

$$s(p_i, t_i) \geq \delta b(0, t_i) \tag{2.9.2}$$

for some $\delta > 0$. Define the rescaled metrics

$$g_i = \frac{1}{b^2(p_i, t_i)} g\left(t + t_i b^2(p_i, t_i)\right), \quad t \in [-\Delta t, 0].$$

For sufficiently small $\Delta t > 0$ the conditions of Proposition 2.8.3 are satisfied and hence $\left(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i\right)$ subsequentially converges to a Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$, $t \in [-\Delta t, 0]$, on which by construction

$$b(p_\infty, 0) = 1.$$

and

$$T_1(p_\infty, 0) = \inf_{\mathcal{C}_\infty \times [-\Delta t, 0]} T_1 = \iota < 0, \tag{2.9.3}$$

as T_1 is a scale invariant quantity. Since $T_s(p_\infty, 0) = 0$, we see from the evolution equation (2.5.11) of T_1 that

$$\begin{aligned}
\partial_t T_1\Big|_{(p_\infty, 0)} &= L[T_1] + \frac{1}{b^2}\left[-4\left(1 + Q^2\right) y^2 + 8Q\left(1 - 2Q^2\right) y + 16Q^2\left(1 - Q^2\right)\right] \\
&\quad + \frac{2yT_1}{b^2}\left(2Q - y\right) \\
&\geq (T_1)_{ss} + \frac{4Q^2}{b^2}\left(1 - Q^2\right) + \frac{2yT_1}{b^2}\left(2Q - y\right),
\end{aligned}$$

where we bounded the zeroth order term from below as in the proof of Lemma 2.5.8. Hence

$$\partial_t T_1\Big|_{(p_\infty,0)} > 0$$

unless

$$\text{case b)}: \quad Q(p_\infty,0) = 0 \text{ and } y(p_\infty,0) = 0$$

or

$$\text{case a)}: \quad Q(p_\infty,0) = 1 \text{ and } y(p_\infty,0) = 0$$

However by (2.9.3) we have

$$\partial_t T_1\Big|_{(p_\infty,0)} \leq 0.$$

showing that either case a) or case b) must hold. We now show that both of these cases are impossible, thereby arriving at a contradiction. First note that by (2.9.2) we know that p_∞ does not lie on the non-principal orbit S^2_o. Therefore the strong maximum principle applied to the evolution equation (2.5.1) of Q shows that in case a) $Q = 0$ everywhere on $\mathcal{C}_\infty \times [-\Delta t, 0]$. This, however, contradicts the non-collapsedness of $\mathcal{C}_\infty$ and therefore case a) cannot occur. In case b) the same argument shows that $Q = 1$ everywhere on $\mathcal{C}_\infty \times [-\Delta t, 0]$. Then applying the strong maximum principle to the evolution equation (2.5.8) of Qy, which simplifies when $Q = 1$, shows that $y = 0$ everywhere on $\mathcal{C}_\infty \times [-\Delta t, 0]$. This, however, implies that $T_1 = 1 > 0$ on $\mathcal{C}_\infty \times [-\Delta t, 0]$ contradicting our assumption that $\iota < 0$.

It remains to be shown that $T_1(p,t) = 0$ if, and only if, $k = 2$ and p lies on the non-principal orbit S^2_o. We argue by contradiction. Assume there exists a point (p,t) in spacetime such that $p \notin S^2_o$ and

$$T_1(p,t) = 0.$$

Then arguing as above, we see that either case a) or case b) must hold true, both of which lead to the same contradiction.

By the same method we may prove that $T_2 \geq 0$ and $T_3 \geq 0$ on $M_k \times (-\infty, 0]$. Note that the evolution equations (2.5.13) and (2.5.14) show that T_2 and T_3 cannot attain a negative infimum, leading to the desired contradiction. $\square$

Next we prove Theorem 2.9.2.

Proof of Theorem 2.9.2. Recall that by Corollary 2.7.6 there exists a $C_1 > 0$ such that $|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{0}$ on $M_2 \times (-\infty, 0]$. Also recall Lemma 2.4.1, which states that $(M_2, g(t)$, $t \in (-\infty, 0]$, is homothetic to the Eguchi-Hanson space if, and only if,

$$x = y = 0 \ \text{ on } \ M_2 \times (-\infty, 0].$$

Therefore it suffices to show that $x = 0$. We follow the proof strategy of Theorem 2.9.1 and argue by contradiction. Assume

$$\iota := \inf_{M_2 \times (\infty, 0]} x < 0$$

and take a sequence of points (p_i, t_i) in spacetime such that

$$x(p_i, t_i) \to \iota.$$

Note that $\iota > -\infty$ as $g(t) \in \mathcal{I}$ for $t \in (-\infty, 0]$. From Lemma 2.9.3 it follows that

$$s(p_i, t_i) \geq \delta b(0, t_i) \tag{2.9.4}$$

for some $\delta > 0$. Define the rescaled metrics

$$g_i = \frac{1}{b^2(p_i, t_i)} g\left(t + t_i b^2(p_i, t_i)\right), \quad t \in [-\Delta t, 0],$$

where $\Delta t > 0$ is chosen such that the conditions of Proposition 2.8.3 are satisfied. Then $(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i)$ subsequentially converges to a Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$, $t \in [-\Delta t, 0]$, on which by construction

$$x(p_\infty, 0) = \iota < 0,$$

since x is a scale-invariant quantity. Furthermore, we see by (2.9.4) that p_∞ does not lie on the non-principal orbit S^2_o. The evolution equation (2.5.7) for x in the Kähler case $y = 0$ simplifies to

$$\partial_t x = L[x] - \frac{4Q^2}{b^2} x$$

which implies that

$$\partial_t x\Big|_{(p_\infty, 0)} = x_{ss} - \frac{4Q^2}{b^2} x > 0$$

unless $Q(p_\infty, 0) = 0$. This, however, cannot happen, as otherwise the strong maximum principle applied to the evolution equation (2.5.1) of Q would imply that $Q = 0$ on $\mathcal{C}_\infty \times [-\Delta t, 0]$. Hence we have arrived at a contradiction and conclude

$$x \geq 0 \ \text{ on } M_2 \times (-\infty, 0].$$

By the same argument one shows that

$$x \leq 0 \ \text{ on } M_2 \times (-\infty, 0].$$

as well, which concludes the proof. □

2.10 Eguchi-Hanson and a family of Type II singularities

In this section we show that Ricci flow solutions $(M_k, g(t))$, $k \geq 2$, starting from a large class of initial metrics encounter a Type II singularity in finite time at the origin. In the case $k = 2$ we show that the Eguchi-Hanson metric can occur as a blow-up limit. Below we state the precise result:

Theorem 2.10.1 (Type II singularities). *Let $(M_k, g(t))$, $k \geq 2$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ (see Definition 2.7.2) with*

$$\sup_{p \in M_2} b(p, 0) < \infty. \tag{2.10.1}$$

Then $g(t)$ encounters a Type II curvature singularity in finite time $T_{sing} > 0$ and

$$\sup_{0 \leq t < T_{sing}} (T_{sing} - t)\, b^{-2}(o, t) = \infty.$$

Furthermore, there exists a sequence of times $t_i \to T_{sing}$ such that the following holds: Consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(o, t_i)} g\left(t_i + b^2(o, t_i) t\right), \quad t \in \big[-b(o, t_i)^{-2} t_i, b(o, t_i)^{-2}\left(T_{sing} - t_i\right)\big).$$

Then $(M_k, g_i(t), o)$ subsequentially converges, in the pointed Gromov-Cheeger sense, to an eternal Ricci flow $(M_k, g_\infty(t), o)$, $t \in (-\infty, \infty)$. When $k = 2$ the metric $g_\infty(t)$ is stationary and homothetic to the Eguchi-Hanson metric.

We do not study the detailed geometry of the singularity models of the Type II singularities arising in the $k \geq 3$ case, however, as stated in Conjecture 4 in the introduction, the author believes that these singularities are modeled on the non-collapsed steady Ricci solitons found in Chapter 1. The author, in collaboration with Jon Wilkening, has carried out numerical simulations supporting this conjecture. A paper summarizing the results is in preparation [AW19].

Outline of proof. Here we sketch the proof of Theorem 2.10.1. First we show in Lemma 2.10.5 that the condition (2.10.1) forces a Ricci flow solution $(M_k, g(t))$, $k \geq 2$, to develop a singularity in finite time $T_{sing} > 0$ at the origin. Then we take a sequence of times $t_i' \to T_{sing}$ and define the rescaled metrics

$$g_i'(t) = \frac{1}{b^2(0, t_i')} g\left(t_i' + b^2(0, t_i') t\right).$$

These metrics subsequentially converge to a singularity model $(M_k, g_\infty'(t))$, $-\infty < t \leq 0$ — an ancient solution of the Ricci flow. Now recall the dichotomy between Type I and Type II singularities and that every Type I singularity is modeled on a shrinking Ricci soliton [EMT11]. Therefore we can prove that the singularity is of Type II by showing that $(M_k, g_\infty'(t))$ is not a shrinking Ricci soliton. For this we apply Theorem 2.6.1, which excludes shrinking solitons whenever (i) $\sup |b_s| < \infty$, (ii) $T_1 > 0$ for $s > 0$ and (iii) $Q \leq 1$ hold. By definition, every metric in $\mathcal{I}$ satisfies conditions (i) and (iii). As these conditions are scale-invariant, they pass to the blow-up limit $(M_k, g_\infty'(t))$. From Theorem 2.9.1 it follows that condition (iii) holds true as well, allowing us to conclude that $(M_k, g_\infty'(t))$ is not a shrinking soliton and that the singularity is of Type II. By the work of Hamilton we can then choose a sequence of times $t_i \to T_{sing}$, possibly different from the sequence t'_i, such that the corresponding blow-ups around the origin converge to an eternal Ricci flow $(M_k, g_\infty(t))$, $-\infty < t < \infty$.

In the $k = 2$ case we show that $(M_2, g_\infty(t))$ is stationary under Ricci flow and homothetic to the Eguchi-Hanson space. What makes $k = 2$ special is that the second term of the right hand side of

$$\partial_t b(0,t) = 2\left(y_s + \frac{k-2}{b}\right)$$

is zero and therefore

$$\partial_t b(0,t) = 2y_s(0,t) \leq 0, \tag{2.10.2}$$

as $y \leq 0$ with equality at S_0^2 for metrics in $\mathcal{I}$. It turns out that for the specific choice of $t_i \to T_{sing}$ from Hamilton's trick we have that on $(M_2, g_\infty(t))$ at S_o^2 at time 0 we have

$$\partial_t b(0,t) = 2y_s(0,t) = 0.$$

An application of L'Hôpital's Rule shows that on S_o^2 we have

$$\frac{y}{Q} = \frac{y_s}{k}.$$

Therefore we can apply the strong maximum principle of Theorem 2.3.2, Case 2, to the evolution equation (2.13.2) of $\frac{y}{Q}$ to show that $y = 0$ everywhere. From Theorem 2.9.2 it then follows that $(M_2, g_\infty(t))$ is homothetic to the Eguchi-Hanson space.

Remark 2.10.2. A priori it may be possible that other sequences of times give rise to blow-up limits around o that are not homothetic to the Eguchi-Hanson space. However in section 2.11 we show that the Eguchi-Hanson space is in fact the unique blow-up limit.

Recap of some properties of singular Ricci flow solutions. Before proving Theorem 2.10.1 we summarize some properties of curvature blow-up rates of Ricci flows encountering singularities and their respective singularity models. For this let $(M, g(t))$, $t \in [0, T)$ be a Ricci flow encountering a singularity at time T. Let

$$K_{max}(t) := \sup_M |Rm_{g(t)}|_{g(t)}.$$

By Shi's result [Shi89] on the short time existence of Ricci flow we have

$$\limsup_{t \nearrow T} K_{max}(t) = \infty.$$

In fact one can show with help of the evolution equation of $|Rm_{g(t)}|^2_{g(t)}$ that

$$\sup_M |Rm_{g(t)}|_{g(t)} \geq \frac{1}{8}\frac{1}{T-t}. \tag{2.10.3}$$

Hamilton [Ham95] introduced the notion of Type I and Type II Ricci flows, which are defined by the rate at which the curvature blows up as $t \nearrow T$. In particular, $(M_2, g(t))$ is of Type I if it satisfies if there exists a $C > 0$ such that for $t \in [0, T)$

$$K_{max}(t) \leq \frac{C}{T-t},$$

In the case that such a constant $C > 0$ does not exists, that is

$$\sup_{t \in [0,T)} (T-t) K_{max}(t) = \infty,$$

we say the singularity is of Type II.

By the work of Naber [N10] and Enders, Müller and Topping [EMT11] every Type I singularity model is a non-flat Ricci shrinking soliton. Hamilton showed how for Type II singularities one can extract a blow-up sequence converging to an eternal Ricci flow [Ham95, Theorem 16.4]. However it remains to be understood whether or not all Type II singularity models are steady solitons. So far all known examples are.

Below we recap the main result of [EMT11]: First note the following definition:

Definition 2.10.3 (see [EMT11, Definition 1.2])**.** A spacetime sequence (p_i, t_i) with $p_i \in M$ and $t_i \nearrow T$ in a Ricci flow is called an **essential blow-up sequence** if there exists a constant $c > 0$ such that

$$|Rm_{g(t_i)}|_{g(t_i)}(p_i) \geq \frac{c}{T-t}.$$

A point $p \in M$ in a Type I Ricci flow is called a (general) **Type I singular point** if there exists an essential blow-up sequence with $p_i \to p$ on M.

Now we state the main result of [EMT11], asserting that Type I singularities are modeled on shrinking Ricci solitons.

Theorem 2.10.4 (see [EMT11, Theorem 1.4])**.** *Let $(M, g(t))$ be a Type I Ricci flow on $[0, T)$ and suppose p is a Type I singular point as in Definition 2.10.3. Then for every sequence $\lambda_j \to \infty$, the rescaled Ricci flows $(M, g_j(t), p)$ defined on $[\lambda_j T, 0)$ by*

$$g_j(t) := \lambda_j g\left(T + \frac{t}{\lambda_j}\right)$$

subconverge to a non-flat gradient shrinking soliton.

We use Theorem 2.10.4 to exclude Type I singularities for Ricci flows satisfying the assumptions of Theorem 2.10.1.

Proof of the main theorem. First we show that a singularity must occur in finite time:

Lemma 2.10.5. *The maximal extension of a Ricci flow* $(M_k, g(t))$, $k \geq 1$, *starting from an initial metric* $g(0) \in \mathcal{I}$ *with*

$$\sup_{p \in M_k} b(p, 0) < \infty$$

encounters a singularity at the S^2_o *in finite time* $T_{sing} > 0$.

Proof. By Shi's short time existence of Ricci flow [Shi89] we have $T_{sing} > 0$. From the evolution equation (2.2.12) of b under Ricci flow it follows that

$$\partial_t b^2 \leq \Delta_{g(t)} b^2 - 4,$$

where we used expression (2.2.4) of the Laplacian. By the maximum principle (see for instance [ChII, Theorem 12.14]) we see that there exists a $T < \infty$ such that

$$\inf_{p \in M_k} b^2(p, t) \to 0 \quad \text{as } t \to T.$$

As $b_s \geq 0$ we conclude that

$$\lim_{t \to T} b(o, t) = 0.$$

From Lemma 2.3.3 it follows that the curvature at S^2_o blows up as $t \to T$. Hence $T = T_{sing}$. □

Below we prove Theorem 2.10.1.

Proof of Theorem 2.10.1. By Lemma 2.10.5 the Ricci flow becomes singular in finite time $T_{sing} > \delta > 0$ and $b(o, t) \to 0$ as $t \nearrow T_{sing}$. Recall that by Theorem 2.7.5 there exist a $C_1 > 0$ such that

$$|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{b^2} \ \text{ on } \ M_k \times [0, T_{sing}).$$

Moreover, by Theorem 2.7.3 there exist constants $\kappa, \rho > 0$ such that $g(t)$ is κ-non-collapsed at scales less or equal to ρ.

Now take a sequence of times $t'_i \nearrow T_{sing}$ such that

1. $b(o, t'_i) \to 0$ as $i \to \infty$
2. $b(o, t) \geq b(o, t'_i)$ for $t \leq t'_i$

Claim 1: The sequence of points (o, t'_i) in spacetime is an essential blow-up sequence.

Proof of Claim: We argue by contradiction. Assume, after passing to a subsequence, that

$$(T_{sing} - t'_i)|Rm_{g(t'_i)}|_{g(t'_i)} \to 0 \text{ as } i \to \infty.$$

Then by Lemma 2.3.3 and the fact that $b_s \geq 0$ for metrics in $\mathcal{I}$ we have

$$b^2(p, t) \geq b^2(o, t) = \frac{4}{R_{2323}} \geq \frac{4}{|Rm_{g(t'_i)}|_{g(t'_i)}(o)},$$

where we used the expression for the curvature component R_{2323} derived in section 2.2. This shows that

$$|Rm_{g(t_i')}|_{g(t_i')}(p) \leq \frac{C_1}{b^2(p,t_i')} \leq \frac{C_1}{4}|Rm_{g(t_i')}|_{g(t_i')}(o) \text{ for } p \in M_k.$$

Therefore

$$\lim_{i\to\infty}(T_{sing} - t_i') \sup_{p\in M_2} |Rm_{g(t_i')}|_{g(t_i')}(p) = 0,$$

which contradicts (2.10.3). This proves the claim. ■

Define the rescaled metrics

$$g_i'(t) = \frac{1}{b^2(0,t_i')} g\left(t_i' + b^2(0,t_i')t\right), \quad t \in [-b^{-2}(o,t_i')t_i', 0],$$

By property (2) above and the fact that $b_s \geq 0$ for metrics in $\mathcal{I}$ it follows that

$$|Rm_{g_i'(t)}|_{g_i'(t)} \leq C_1 \text{ on } M_k \times [-b^{-2}(o,t_i')t_i', 0].$$

Note also that the rescaled metrics $g_i'(t)$ are κ-non-collapsed at scales tending to infinity. Corollary 2.8.2 then implies that $(M_k, g_i'(t), o)$ subsequentially converges, in the Cheeger-Gromov sense, to an ancient Ricci flow $(M_\infty, g_\infty'(t), o)$, $t \in (-\infty, 0]$, where $M_\infty \cong M_k$. By Theorem 2.9.1 we have

$$T_1(p,t) > 0 \text{ on } M_\infty \setminus S_o^2 \times (-\infty, 0]$$

on the blow-up limit $g_\infty'(t)$. Theorem 2.6.1 shows that $g_\infty'(t)$ cannot be a shrinking soliton, which by the contrapositive of Theorem 2.10.4 proves that the singularity is of Type II. Therefore

$$\sup_{M\times[0,T_{sing})} (T_{sing} - t)\,|Rm_{g(t)}|_{g(t)} = \infty$$

from which we see that

$$\sup_{t\in[0,T_{sing})} (T_{sing} - t)\, b^{-2}(0,t) = \infty.$$

Now we mimic the proof of [Ham95, Theorem 16.4, Type II(a)] to construct an eternal blow-up limit. Pick a sequence of times $T_i < T_{sing}$ satisfying

$$(T_{sing} - T_i)\, b^{-2}(o,t) \to \infty$$

as $i \to \infty$. Then we can choose $t_i < T_i$ such that

$$(T_i - t_i)\, b^{-2}(o,t_i) = \sup_{t\leq T_i} (T_i - t)\, b^{-2}(o,t) \tag{2.10.4}$$

as the latter goes to zero as $t \to T_i$. Consider the rescaled Ricci flow solutions

$$g_i(t) = \frac{1}{b^2(0,t_i)} g\left(t_i + b^2(0,t_i)t\right),$$

which exist for $-A_i \leq t \leq B_i$ with

$$A_i = t_i b^{-2}(o, t_i) \to \infty$$
$$B_i = (T_i - t_i)\, b^{-2}(o, t_i) \to \infty.$$

If we write a_i, b_i for the warping functions of the rescaled metric $g_i(t)$ we obtain from equation (2.10.4) the following inequality

$$(B_i - t)\, b_i^{-2}(0, t) \leq B_i.$$

Note that here t is the time variable of the rescaled Ricci flow $g_i(t)$. Therefore for any fixed t we have

$$b_i^{-2}(o, t) \leq \frac{B_i}{B_i - t} \to 1 \quad \text{as } i \to \infty \tag{2.10.5}$$

and

$$b_i^{-2}(o, 0) = 1. \tag{2.10.6}$$

From this, the fact that $b_s \geq 0$ and the curvature bound of Theorem 2.7.5, we see that on bounded time intervals the curvatures of $g_i(t)$ eventually become bounded by $2C_1$. In addition to this the metrics $g_i(t)$ are κ-non-collapsed at larger and larger scales. Therefore Corollary 2.8.2 implies that $(M_2, g_i(t), o)$ subsequentially converges to an eternal Ricci flow $(M_2, g_\infty(t), o)$. Furthermore (2.10.5) and (2.10.6) show that that

$$b_\infty(o, t) \geq 1 \ \text{ for } \ t \in (-\infty, \infty) \tag{2.10.7}$$

and

$$b_\infty(o, 0) = 1, \tag{2.10.8}$$

where we write a_∞ and b_∞ for the warping functions of the metric $g_\infty(t)$. Notice that (2.10.7) and (2.10.8) imply that at time 0 on S_o^2 we have

$$\partial_t b_\infty = 2\left((y_\infty)_s + \frac{k-2}{b_\infty}\right) = 0, \tag{2.10.9}$$

where $y_\infty = (b_\infty)_s - \frac{a_\infty}{b_\infty}$ corresponds to the Kähler quantity y on the g_∞ background.

Now it only remains to be shown that in the $k = 2$ case $g_\infty(t)$ is stationary and homothetic to the Eguchi-Hanson metric. In the following we drop the ∞ subscript and let a, b, Q, y be with respect to the metric $g_\infty(t)$. Note that equation (2.10.9) and an application of L'Hôpital's Rule show that at time 0 on S_o^2 we have

$$y_s = \frac{y}{Q} = 0$$

The evolution equation for $\frac{y}{Q}$ derived in the Appendix A is

$$\partial_t \left(\frac{y}{Q}\right) = \left(\frac{y}{Q}\right)_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)\left(\frac{y}{Q}\right)_s + \frac{2}{b^2}\frac{y}{Q}\left(2 + \frac{y}{Q}\right)(Qb_s - 2a_s). \tag{2.10.10}$$

Because $g_\infty(t) \in \mathcal{I}$ is of bounded curvature we see that

$$\frac{1}{b^2}\left(2+\frac{y}{Q}\right)(Qb_s - 2a_s) = \frac{1}{b^2}\left(-\frac{2a_sb_s}{Q} - 2a_s + Qb_s + b_s^2\right)$$

is bounded. Note that we applied Lemma 2.3.3 to show that $\frac{1}{b^2}$ is bounded. Therefore we may apply the strong maximum principle of Theorem 2.3.2, Case 2, to deduce that

$$\frac{y}{Q} = 0 \text{ on } M_2 \times (-\infty, 0],$$

yielding that $g_\infty(t)$ is Kähler in the the $k = 2$ case. By Theorem 2.9.2 we then deduce that $g_\infty(t)$ is homothetic to the Eguchi-Hanson metric, which proves the desired result. □

2.11 Ancient Ricci flows Part II: $k = 2$ case

In this section we prove that every non-collapsed ancient Ricci flow in the class of metrics $\mathcal{I}$ is isometric to the Eguchi-Hanson metric:

Theorem 2.11.1 (Unique ancient flow)**.** *Let $\kappa > 0$ and $(M_2, g(t))$, $t \in (-\infty, 0]$, be an ancient Ricci flow that is κ-non-collapsed at all scales and $g(t) \in \mathcal{I}$, $t \in (-\infty, 0]$ (see Definition 2.7.2). Then $g(t)$ is stationary and homothetic to the Eguchi-Hanson metric.*

An immediate consequence of this theorem is that for *every* sequence of times $t_i \to T_{sing}$ in Theorem 2.10.1, the rescaled Ricci flows

$$g_i(t) = \frac{1}{b^2(o, t_i)} g\left(t + t_i b^2(o, t_i)\right), \quad t \in \left[-b(o, t_i)^{-2} t_i, 0\right],$$

subsequentially converge to the Eguchi-Hanson space. In other words, the Eguchi-Hanson space is the *unique* limit of blow-ups around the origin. With a little extra work one can show the slightly more general result, asserting that blow-up limits centered at points close to, but not necessarily on the tip of M_2, subsequentially converge to the Eguchi-Hanson space:

Corollary 2.11.2. *Let $(M_2, g(t))$, $t \in [0, T_{sing})$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ (see Definition 2.7.2) that develops a singularity at time T_{sing}. Let (p_i, t_i) be a sequence of points in spacetime with $t_i \to T_{sing}$ satisfying*

$$\sup_i \frac{b(p_i, t_i)}{b(o, t_i)} < \infty$$

and consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(p_i, t_i)} g\left(t_i + b^2(p_i, t_i) t\right), \quad t \in \left[-t_i b^{-2}(p_i, t_i), 0\right].$$

Then $(M_2, g_i(t), p_i)$ subsequentially converges, in the Gromov-Cheeger sense, to a blow-up limit $(M_2, g_\infty(t), p_\infty)$, $t \leq 0$, which is homothetic to the Eguchi-Hanson space.

We defer the proof of Corollary 2.11.2 to the end of subsection 2.11.

Outline of Proof

Here we outline the proof of Theorem 2.11.1. Below we take $(M_2, g(t))$, $t \in (-\infty, 0]$, to be a non-collapsed ancient Ricci flow with $g(t) \in \mathcal{I}$, $t \in (-\infty, 0]$. We construct a continuously varying one-parameter family of functions

$$f_\theta : [0,1] \to [0,1], \quad \theta \in (0,1],$$

satisfying the following five requirements:

1. For every $\theta \in (0,1]$ the condition

$$Z_\theta(\xi, t) := \frac{x}{Q^2} + f_\theta(Q) = \frac{a_s + Q^2 - 2}{Q^2} + f_\theta(Q) \geq 0$$

   is preserved on the $(M_2, g(t))$ background.

2. For every $0 \leq Q < 1$

$$f_\theta(Q) \longrightarrow 0 \text{ as } \theta \longrightarrow 0$$

3. For every $\theta \in (0,1]$ there exists a $Q_\theta \in [0,1)$ such that

$$f(Q) < 1 \text{ for } Q < Q_\theta,$$

   and

$$f(Q) = 1 \text{ for } Q \geq Q_\theta.$$

   Furthermore Q_θ depends continuously on θ.

4. For $\theta = 1$

$$f_1 = 1$$

   everywhere.

5. For every $\theta \in (0,1]$ the function f_θ is extendable to a smooth even function around 0.

Remark 2.11.3. We briefly remark on some of the properties of f_θ:

- In the expression for Z_θ of requirement (1) we take x, Q and a_s to be functions of spacetime. For brevity we do not express the dependence explicitly.

- The term $\frac{x}{Q^2}$ can be extended smoothly to the non-principal orbit S^2_o, as $x = x_s = 0$ at $s = 0$. Therefore Z_θ is well-defined on M_2.

- When $\theta = 1$ we already know that

$$Z_1 = \frac{x + Q^2}{Q^2} \geq 0$$

  in $M_2 \times (-\infty, 0]$, as from Theorem 2.9.1 it follows that $T_1 = Q^2 Z_1 \geq 0$ on $M_2 \times (-\infty, 0]$.

- At any point (p,t) in spacetime such that $Q(p,t) \geq Q_\theta$ we have

$$Z_\theta(p,t) = Z_1(p,t) \geq 0.$$

- The family $f_\theta(Q)$, $\theta \in (0,1]$, we construct below is smooth everywhere apart from when $Q = Q_\theta$. It will become clear later that this does not pose a problem.

In subsection 2.11 we show that at points (p,t) in spacetime at which $Q(p,t) \neq Q_\theta$, or equivalently at points where f is smooth, the evolution equation of the corresponding Z_θ can locally be written as

$$\partial_t Z_\theta = [Z_\theta]_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)[Z_\theta]_s + \frac{1}{b^2}\left(W_\theta + Z_\theta \tilde{D}_\theta\right),$$

where W_θ and $\tilde{D}_\theta$ are bounded and scale-invariant expressions involving b_s, Q, $f_\theta(Q)$, $f'_\theta(Q)$ and $f''_\theta(Q)$. Again, all quantities in the evolution equation of Z_θ should be interpreted as functions of spacetime.

In subsection 2.11 we construct a family of functions f_θ, $\theta \in (0,1]$, by solving an initial value problem for a second order non-linear ordinary differential equation. Subsequently we show that the family satisfies requirements (1)-(5) listed above. In particular, we show in subsection 2.11 that for the constructed family — on a non-collapsed ancient Ricci flow background — the following property holds true: For all points (p,t) in spacetime such that $Q(p,t) < Q_\theta$, we have $W_\theta(p,t) \geq 0$, $\theta \in (0,1]$. This fact, in conjunction with the fourth bullet point of Remark 2.11.3, essentially shows that for each $\theta \in (0,1]$ the inequality $Z_\theta \geq 0$ is preserved on the $g(t)$ background.

Once we have shown that our family f_θ, $\theta \in (0,1]$, satisfies requirements (1)-(5), we will use a blow-up argument in conjunction with the strong maximum principle to show that if for some $\theta_0 \in (0,1]$ the inequality $Z_\theta \geq 0$ holds for all $\theta \in [\theta_0, 1]$, then there must exists an $\theta_1 < \theta_0$ such that the inequality also holds for all $\theta \in [\theta_1, 1]$. This shows that the set

$$\mathcal{E} = \left\{\theta \in (0,1] \,\middle|\, Z_{\theta'} \geq 0 \text{ for all } \theta \leq \theta' \leq 1\right\} \subseteq (0,1]$$

is open. As $\mathcal{E}$ is defined by a closed condition and therefore closed, it follows that $\mathcal{E} = (0,1]$ and therefore

$$Z_\theta \geq 0 \quad \text{for all } \theta \in (0,1].$$

By property (2) of f_θ we deduce that at all points (p,t) in spacetime such that $Q(p,t) < 1$ we have

$$x(p,t) \geq 0.$$

Note that by the strong maximum principle applied to the evolution equation (2.5.1) of Q it follows that $Q < 1$ and hence $x \leq 0$ everywhere. Now recall Theorem 2.9.1 which states that

$$x \leq Qy \leq 0 \ \text{ on } \ M_2 \times (-\infty, 0].$$

Therefore

$$x = y = 0 \ \text{ on } \ M_2 \times (-\infty, 0]$$

and we conclude that the metric $g(t)$ is homothetic to the Eguchi-Hanson metric by Lemma 2.4.1.

Evolution equations

The main difficulty in carrying out the proof is to find a family of functions f_θ, $\theta \in (0, 1]$, for which requirement (1) is satisfied. Our strategy is to first derive the evolution equation of Z_θ for a general f_θ and then reduce the problem to solving a second order ordinary differential equation in f_θ. For this, first note that the evolution equation of $f_\theta(Q)$ away from the non-principal orbit S^2_o can be written as

$$\partial_t f_\theta(Q) = [f_\theta(Q)]_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)[f_\theta(Q)]_s + \frac{1}{b^2}C_f,$$

where

$$C_f = \left(8a_sb_s - 3\frac{a_s^2}{Q} - 5Qb_s^2 + 4Q\left(1 - Q^2\right)\right)f' - \left(a_s - Qb_s\right)^2 f'' \tag{2.11.1}$$

The computation is carried out in the Appendix A.

Remark 2.11.4. Some remarks on the evolution equation of $f_\theta(Q)$:

- We often omit the dependence of our quantities on spacetime, i.e. by $f_\theta(Q)$ we mean $f_\theta(Q(p, t))$.
- For brevity we often omit the dependence of f on θ and Q, as in the expression for C_f above. For instance, we write f' for $f'_\theta(Q)$ and f'' for $f''_\theta(Q)$.
- Note that by Lemma 2.2.2 the quantity $Q = \frac{a}{b}$ as a function of s can be extended to an odd function around the origin. Therefore as long as $f_\theta : [0, 1] \to [0, 1]$ is extendable to an even function around the origin, the term $\frac{f'}{Q}$ and hence C_f can be smoothly extended to all of M_2.

From equation (2.11.1) and the evolution equations (2.5.3) and (2.5.5) of a_s and Q^2, respectively, we see that the evolution equation of Z_θ can be written as

$$\partial_t Z_\theta = [Z_\theta]_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)[Z_\theta]_s + \frac{1}{b^2}\left(C_{Z,0} + C_{Z,1}Z_\theta + +C_{Z,2}Z_\theta^2\right), \tag{2.11.2}$$

after having eliminated any occurring a_s by substituting

$$a_s = Q^2 Z_\theta - fQ^2 - Q^2 + 2.$$

A computation carried out in the Appendix A shows that

$$C_{Z,0} = A_0 + \left[\frac{b_s}{Q}\right] A_1 + \left[\frac{b_s}{Q}\right]^2 A_2,$$

where

$$\begin{aligned} A_0 =& - Q^4 f^2 f'' - 2Q^4 f f'' - Q^4 f'' + 4Q^2 f f'' + 4Q^2 f'' - 4f'' - 3Q^3 f^2 f' \\ & - 6Q^3 f f' - 7Q^3 f' + 12Q f f' + 16Q f' - \frac{12f'}{Q} - 2Q^2 f + 8f - 2Q^2 - 4 \\ A_1 =& - 2Q^4 f f'' - 2Q^4 f'' + 4Q^2 f'' - 8Q^3 f f' - 8Q^3 f' \\ & + 16Q f' - 4Q^2 f^2 - 8Q^2 f + 8f + 4Q^2 + 8 \\ A_2 =& - Q^4 f'' - 5Q^3 f' - 2Q^2 f - 2Q^2 - 4 \end{aligned}$$

Similarly, we compute the expressions for $C_{Z,1}$ and $C_{Z,2}$ in the Appendix A, however their exact forms are not important for our analysis. It is only important to note that when f is extendable to an even function around 0, the quantities $C_{Z,i}$, A_i, $i = 0, 1, 2$, are scale-invariant, bounded, and can be extended smoothly to S_o^2.

For reasons that will become clear below, we rewrite the equation (2.11.2) in the form

$$\partial_t Z_\theta = [Z_\theta]_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)[Z_\theta]_s + \frac{1}{b^2}\left(W_\theta + Z_\theta D_\theta\right), \tag{2.11.3}$$

where

$$W_\theta = A_0 + \left[\frac{b_s}{Q} - Z_\theta\right] A_1 + \left[\frac{b_s}{Q} - Z_\theta\right]^2 A_2 \tag{2.11.4}$$

and

$$D_\theta = C_{Z,1} + ZC_{Z,2} + A_1 - A_2\left(Z_\theta - 2\frac{b_s}{Q}\right)$$

Sometimes it will be useful to regard W_θ as a quadratic polynomial. Therefore we define

$$w_\theta(z) = A_0 + A_1 z + A_2 z^2$$

Then

$$W_\theta = w_\theta\left(\frac{b_s}{Q} - Z_\theta\right).$$

In the proof of Theorem 2.11.1 we also need the evolution equation of

$$Z_1 = \frac{x}{Q^2} + 1,$$

which can be written as

$$\partial_t Z_1 = [Z_1]_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)[Z_1]_s + \frac{1}{b^2}\left(C_{Z_1,0} + C_{Z_1,1}Z_1 + C_{Z_1,2}Z_1^2\right) \tag{2.11.5}$$

where

$$C_{Z_1,0} = \frac{1}{Q^2}\left(-4\left(1+Q^2\right)y^2 + 8Q\left(1-2Q^2\right)y + 16Q^2\left(1-Q^2\right)\right)$$

and $C_{Z_1,1}$, $C_{Z_1,2}$ are a bounded scale-invariant functions of a_s, b_s and Q. The derivation of this evolution equation is carried out in the Appendix A. Note the following lemma:

Lemma 2.11.5. *Let $(M_2, g(t))$, $t \in (-\infty, 0]$, be an ancient Ricci flow as in Theorem 2.11.1. Then*

$$Z_1 \geq 0$$

and

$$C_{Z_1,0} \geq 4\left(1-Q^2\right)$$

everywhere in $M_2 \times (-\infty, 0]$.

Proof. By Theorem 2.9.1 we know that $Z_1 = \frac{T_1}{Q^2} \geq 0$ in $M_2 \times (-\infty, 0]$. Moreover, notice the similarity of $C_{Z_1,0}$ to the zeroth order term in the evolution equation (2.5.11) of T_1. Therefore we see by the proof of Lemma 2.5.8 that $C_{Z_1,0} \geq 4\left(1-Q^2\right)$ for metrics in $\mathcal{I}$. □

In the proof of Theorem 2.11.1 we deform the inequality $Z_1 \geq 0$ along a path of conserved inequalities $Z_\theta \geq 0$, $\theta \in (0,1]$. Thus $Z_1 \geq 0$ is the starting point for successively constraining the ancient Ricci flow towards the Eguchi-Hanson space. Below we construct the f_θ leading to the conserved inequalities $Z_\theta \geq 0$.

Construction of f_θ, $\theta \in (0,1]$

The goal of the following discussion is to find a family of functions $f_\theta : [0,1] \to [0,1]$, $\theta \in (0,1]$, such that

$$W_\theta \geq 0$$

is non-negative on ancient Ricci flows satisfying $Z_\theta \geq 0$. For this we consider solutions to the ordinary differential equation

$$\begin{aligned} 0 = -4\left(1-Q^2\right)^2 f'' - 4\left(1-Q^2\right)\left(Q^2 f - 5Q^2 + 3\right)\frac{f'}{Q} \\ + 2f\left(f^2Q^2 + 3fQ^2 - 6f - 6Q^2 + 8\right), \end{aligned} \tag{2.11.6}$$

which is equivalent to

$$w_\theta\left(-f+1\right) = 0. \tag{2.11.7}$$

Note that we are now regarding Q as an independent variable and not as a function of spacetime. Before we explain how we arrived at this differential equation, we list some of its properties below. For clarity of exposition we defer their proofs to subsection 2.11.

Lemma 2.11.6. *For every $f_0 \in \mathbb{R}$ the ordinary differential equation (2.11.6) possesses an even analytic solution around the origin with initial condition*

$$f(0) = f_0.$$

Furthermore, f varies smoothly with f_0.

Lemma 2.11.7. *Let $f : [0, Q_{max}) \to \mathbb{R}$, $Q_{max} \leq 1$, be the maximal solution to the ordinary differential equation (2.11.6) with initial condition $0 < f(0) < 1$. Then on any interval $(0, Q_*)$, $Q_* \leq Q_{max}$, on which $0 < f(Q) \leq 1$ we have $f'(Q) > 0$.*

Lemma 2.11.8. *Let $\theta \in (0,1]$ and $f_\theta : 0 \in I \to \mathbb{R}$ be the maximal solution to the ordinary differential equation (2.11.6) with $f_\theta(0) = \theta$. Then there exists a $Q_\theta \in [0,1)$ such that*

$$f_\theta(Q_\theta) = 1$$

and

$$f_\theta(Q) < 1 \text{ for } 0 \leq Q < Q_\theta.$$

Furthermore,

1. *Q_θ varies continuously with $\theta \in (0,1]$*
2. *$Q_\theta \to 1$ as $\theta \to 0$*
3. *$Q_1 = 0$*

For each $\theta \in (0,1]$ let

$$\phi_\theta : [0, Q_\theta] \to [\theta, 1]$$

be the solution to the differential equation (2.11.6) with initial condition

$$\phi_\theta(0) = \theta$$

and define f_θ, $\theta \in (0,1]$, as follows:

$$f_\theta(Q) = \begin{cases} \phi_\theta(Q) & \text{for } 0 \leq Q \leq Q_\theta \\ 1 & \text{for } Q_\theta < Q \leq 1 \end{cases} \tag{2.11.8}$$

Note that f_θ is continuous but in general not smooth at $Q = Q_\theta$. This is not a problem, as will become clear later. In summary, we have:

Proposition 2.11.9. *There exists a unique continuously varying family of continuous functions $f_\theta : [0,1] \to [0,1]$ and numbers $Q_\theta \in [0,1)$ for $\theta \in (0,1]$ satisfying the following properties:*

- *$f_\theta(Q)$ solves (2.11.6) or equivalently $w_\theta(-f_\theta(Q) + 1) = 0$ for $0 \leq Q \leq Q_\theta$*

- *$f_\theta(0) = \theta$*
- *$f_\theta(Q) < 1$ for $Q < Q_\theta$ and $f_\theta(Q) = 1$ for $Q \geq Q_\theta$*
- *$f_\theta(Q)$ is strictly increasing in Q when $0 < Q < Q_\theta$*
- *$f_\theta(Q)$ is extendable to an even function around the origin*
- *Q_θ varies continuously with θ*
- *$Q_\theta \to 1$ as $\theta \to 0$ and $Q_1 = 0$*
- *For every $Q \in [0, 1)$ we have $f_\theta(Q) \to 0$ as $\theta \to 0$*

Non-negativity of W_θ

For the choice of f_θ, $\theta \in (0, 1]$, defined above the following proposition holds true:

Proposition 2.11.10. *Let $\theta \in (0, 1]$ and f_θ be as defined in (2.11.8). Assume $(M_2, g(t))$, $t \in (-\infty, 0]$, is a non-collapsed ancient Ricci flow with $g(t) \in \mathcal{I}$ for $t \in (-\infty, 0]$ and $Z_\theta \geq 0$ everywhere. Suppose at the point (p, t) in spacetime $Q(p, t) < Q_\theta$. Then*

$$W_\theta(p, t) \geq 0$$

with equality if, and only if,

$$T_2(p, t) = 0.$$

We prove this proposition in multiple steps. First note

Lemma 2.11.11. *Let f_θ, $\theta \in (0, 1]$, be the family of functions as defined in Proposition 2.11.9. Then*

$$A_2 = -Q^4 f'' - 5Q^3 f' - 2Q^2 f - 2Q^2 - 4 < 0 \tag{2.11.9}$$

for $0 \leq Q < Q_\theta$. Thus $w_\theta(z) = A_2 z^2 + A_1 z + A_0$ is concave in z whenever $0 \leq Q < Q_\theta$.

The proof of this technical lemma can be found in subsection 2.11. Furthermore we have

Lemma 2.11.12. *Let $\theta \in (0, 1]$. Let $(M_2, g(t))$, $t \in (-\infty, 0]$, be a non-collapsed ancient Ricci flow with $g(t) \in \mathcal{I}$ for $t \in (-\infty, 0]$ and $Z_\theta \geq 0$ everywhere. Then*

$$-f_\theta(Q) + 1 \leq \frac{b_s}{Q} - Z_\theta \leq \min\left(1, -f_\theta(Q) + \frac{3}{Q^2} - 2\right).$$

and

$$\min\left(1, -f_\theta(Q) + \frac{3}{Q^2} - 2\right) = \begin{cases} 1 & \text{if } f_\theta(Q) \leq 3\frac{1-Q^2}{Q^2} \\ -f + \frac{3}{Q^2} - 2 & \text{otherwise} \end{cases} \tag{2.11.10}$$

Proof. By Theorem 2.9.1 and since $g(t) \in \mathcal{I}$ we know that

$$\begin{aligned} y &= b_s - Q \le 0 \\ T_2 &= Qy - x = -a_s + Qb_s + 2\left(1 - Q^2\right) \ge 0 \\ T_3 &= a_s - Qb_s + 1 - Q^2 \ge 0 \end{aligned}$$

on $M_2 \times (-\infty, 0]$. Therefore $y \le 0$ implies

$$\frac{b_s}{Q} - Z_\theta \le 1 - Z_\theta$$

and $T_2 \ge 0$ implies

$$\frac{b_s}{Q} - Z_\theta = \frac{Qb_s - a_s - Q^2 + 2}{Q^2} - f_\theta(Q) \ge 1 - f_\theta(Q)$$

and finally $T_3 \ge 0$ implies

$$\frac{b_s}{Q} - Z_\theta = \frac{Qb_s - a_s - Q^2 + 2}{Q^2} - f_\theta(Q) \le -f_\theta(Q) + \frac{3}{Q^2} - 2.$$

Now applying the assumption $Z_\theta \ge 0$ proves the desired result. $\square$

Recalling that by definition

$$W_\theta = w_\theta\left(\frac{b_s}{Q} - Z_\theta\right),$$

the above Lemma 2.11.12 and concavity of $w_\theta(z)$ show that to prove Proposition 2.11.10 it suffices to check that for $\theta \in (0, 1]$ and $0 \le Q < Q_\theta$ we have

$$\begin{aligned} &\alpha := w_\theta\left(-f_\theta(Q) + 1\right) \ge 0, \\ &\beta := w_\theta\left(1\right) \ge 0 \ \text{ whenever } \ f_\theta(Q) \le 3\frac{1 - Q^2}{Q^2}, \\ &\gamma := w_\theta\left(-f_\theta(Q) + \frac{3}{Q^2} - 2\right) \ge 0 \ \text{ whenever } \ f_\theta(Q) \ge 3\frac{1 - Q^2}{Q^2}, \end{aligned}$$

where f_θ is as defined in Proposition 2.11.9. Note that γ is only defined for $Q^2 > 0$. This however does not pose a problem as

$$1 \ge f_\theta(Q) \ge 3\frac{1 - Q^2}{Q^2}$$

implies that $Q^2 \ge \frac{3}{4} > 0$. Recall that by the properties of $f_\theta(Q)$ summarized in Proposition 2.11.9 we have

$$w_\theta\left(-f_\theta(Q) + 1\right) = 0 \ \text{ for } \ 0 \le Q < Q_\theta,$$

and therefore only need to investigate the sign of β and γ in their respective regimes. This explains why we chose to define $f_\theta(Q)$ via the ordinary differential equation (2.11.6). In the following technical lemma, the proof of which we defer to subsection 2.11, we show that for our choice of f_θ the functions β and γ are in fact positive in their respective regimes:

Lemma 2.11.13. *Fix $\theta \in (0,1]$ and let $f_\theta(Q)$ as defined in Proposition 2.11.9. Then for $0 \leq Q < Q_\theta$ we have*

$$\beta > 0 \ \text{ whenever } \ f_\theta(Q) \leq 3\frac{1-Q^2}{Q^2},$$

and

$$\gamma > 0 \ \text{ whenever } \ f_\theta(Q) \geq 3\frac{1-Q^2}{Q^2}.$$

Now we can prove Proposition 2.11.10.

Proof of Proposition 2.11.10. By Lemma 2.11.11 and Lemma 2.11.13 we know that $W_\theta \geq 0$ whenever $0 \leq Q < Q_\theta$, with equality if and only if

$$\frac{b_s}{Q} - Z_\theta = 1 - f_\theta(Q),$$

which by the definition of Z_θ is equivalent to $T_2 = 0$. □

Remark 2.11.14. The proof of Proposition 2.11.10 essentially implies that for every $\theta \in (0,1]$ the inequality $Z_\theta \geq 0$ is preserved on Ricci flow backgrounds in $\mathcal{I}$ satisfying $T_1 \geq 0$, $T_2 \geq 0$ and $T_3 \geq 0$. We do not prove this here, as our proof of Theorem 2.11.1 does not rely on this fact.

Proof of main theorem

Next, we prove that the Eguchi-Hanson space is the unique ancient Ricci flow in the class $\mathcal{I}$.

Proof of Theorem 2.11.1. Recall that by Corollary 2.7.6 there exists a $C_1 > 0$ such that

$$|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{b^2}.$$

Moreover, by Theorem 2.9.1

$$T_1, T_2, T_3 \geq 0 \ \text{ on } \ M_2 \times (\infty, 0]$$

and by Lemma 2.11.5

$$Z_1 \geq 0 \ \text{ on } \ M_2 \times (\infty, 0].$$

Hence we may assume that there exists a $\theta_0 \in (0,1]$ such that for all $\theta \in [\theta_0, 1]$

$$Z_\theta = \frac{x}{Q^2} + f_\theta(Q) \geq 0 \ \text{ on } \ M_2 \times (\infty, 0].$$

Claim 1: For every $0 \leq Q_* < 1$ we have

$$\inf \left\{ Z_{\theta_0}(p,t) \,\middle|\, (p,t) \in M_2 \times (\infty, 0] \text{ such that } Q(p,t) \leq Q_* \right\} > 0.$$

Proof of Claim: We argue by contradiction. Assume there exists a sequence of points (p_i, t_i) in spacetime such that

$$Q(p_i, t_i) \leq Q_*$$

and

$$Z_{\theta_0}(p_i, t_i) \to 0 \text{ as } i \to \infty.$$

Consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(p_i, t_i)} g(t_i + b^2(p_i, t_i)t), \quad t \in [-\Delta t, 0].$$

For sufficiently small $\Delta t > 0$ the conditions of Proposition 2.8.3 are satisfied and therefore $(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i)$, $t \in [-\Delta t, 0]$ subsequentially converges to a Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$, $t \in [-\Delta t, 0]$. Write

$$\Omega = \mathcal{C}_\infty \times [-\Delta t, 0].$$

By construction

$$Z_{\theta_0}(p_\infty, 0) = \inf_\Omega Z_{\theta_0} = 0.$$

Now we need to distinguish two cases:

Case 1: $Q(p_\infty, 0) < Q_{\theta_0}$

Then there exists an $r \in (0, \frac{1}{2})$ and $\Delta t' \in (0, \Delta t)$ such that on the parabolic set

$$\Omega' = C_{g_\infty(0)}(p_\infty, r) \times [-\Delta t', 0] \subset \Omega$$

we have $Q < Q_{\theta_0}$. By the strong maximum principle of Theorem 2.3.2 applied to the evolution equation (2.11.3) of Z_{θ_0} we have

$$Z_{\theta_0} = 0 \text{ on } \Omega'$$

and therefore

$$(Z_{\theta_0})_s = (Z_{\theta_0})_{ss} = 0 \text{ on } \Omega'.$$

By the evolution equation (2.11.3) of Z_{θ_0} we see that that

$$0 = \partial_t Z_{\theta_0} = \mathcal{Q}_{\theta_0} \text{ in } \Omega',$$

which by Proposition 2.11.10 implies

$$T_2 = Qy - x = 0 \text{ in } \Omega'.$$

However, the evolution equation (2.5.13) of T_2 then implies

$$y = 0 \text{ on } \Omega'.$$

and thus also

$$x = 0 \text{ on } \Omega'.$$

That in turn implies

$$Z_{\theta_0}(p_\infty, 0) = f(Q(p_\infty, 0)) \geq \theta_0 > 0,$$

which is a contradiction.

Case 2: $Q(p_\infty, 0) \geq Q_{\theta_0}$

Recall that at points (p,t) in spacetime satisfying $Q(p,t) \geq Q_{\theta_0}$ we have $Z_{\theta_0}(p,t) = Z_1(p,t)$. In this case we therefore have

$$Z_1(p_\infty, 0) = Z_{\theta_0}(p_\infty, 0) = 0.$$

By the strong maximum principle applied to the evolution equation (2.11.5) of Z_1 and Lemma 2.11.5 we deduce

$$Z_1 = 0 \text{ on } \Omega.$$

Furthermore, we see that this is only possible when

$$Q = 1 \text{ on } \Omega.$$

which contradicts

$$Q(p_\infty, 0) \leq Q_* < 1.$$

This concludes the proof of the claim. ■

Thus for every $Q_* \in (0,1)$ there exists a $\delta > 0$ such that for all points (p,t) in spacetime satisfying $0 \leq Q(p,t) \leq Q_*$ we have

$$Z_{\theta_0}(p,t) > \delta.$$

By the continuous dependence of Z_θ and Q_θ on θ, and the fact that $Z_\theta = Z_{\theta'}$ at points (p,t) in spacetime at which $Q(p,t) \geq \max(Q_\theta, Q_{\theta'})$, there exists an $\theta_1 < \theta_0$ such that for $\theta \in [\theta_1, 1]$

$$Z_\theta \geq 0 \text{ on } M_2 \times (-\infty, 0].$$

Now consider the set

$$\mathcal{E} = \left\{\theta \in (0,1] \,\middle|\, Z_{\theta'} \geq 0 \text{ for } \theta \leq \theta' \leq 1\right\} \subseteq (0,1]$$

The above argument shows that $\mathcal{E}$ is an open subset of $(0,1]$. As the condition $Z_\theta \geq 0$ is closed and f_θ depends continuously on θ, it follows that $\mathcal{E}$ is also a closed subset of $(0,1]$. Hence by connectedness of $(0,1]$ it follows that $\mathcal{E} = (0,1]$ and thus for all $\theta \in (0,1]$

$$Z_\theta \geq 0 \text{ on } M_2 \times (-\infty, 0].$$

Note that by the strong maximum principle applied to the evolution equation (2.5.1) of Q

$$Q < 1 \text{ on } M_2 \times (-\infty, 0],$$

as otherwise $Q = 1$ everywhere, which is not true. As $Z_\theta = \frac{x}{Q^2} + f_\theta(Q)$ and by Proposition 2.11.9 for every $0 \leq Q < 1$ we have $f_\theta(Q) \to 0$ as $\theta \to 0$ it follows that

$$x \geq 0 \text{ on } M_2 \times (-\infty, 0].$$

However, as

$$T_2 = Qy - x \geq 0 \text{ on } M_2 \times (-\infty, 0]$$

and $y \leq 0$ by the assumption that $g(t) \in \mathcal{I}$ it follows that

$$x = y = 0 \text{ on } M_2 \times (-\infty, 0].$$

By Lemma 2.4.1 we conclude that $(M_2, g(t)), t \in (-\infty, 0]$, is stationary and homothetic to the Eguchi-Hanson space. □

Now we prove Corollary 2.11.2.

Proof of Corollary 2.11.2. By Theorem 2.7.5 and the fact that $b_s \leq 0$ for metrics in $\mathcal{I}$ there exists a $C_1 > 0$ such that

$$|Rm_{g(t)}|_{g(t)}(p) \leq \frac{C_1}{b^2(p,t)} \leq \frac{C_1}{b^2(o,t)}.$$

This shows that

$$b(o,t) \to 0 \text{ as } t \to T_{sing}.$$

As by assumption

$$C := \sup \frac{b(p_i, t_i)}{b(o, t_i)} < \infty$$

and $y \leq 0$ by the fact that $g(t) \in \mathcal{I}$ it follows that

$$\partial_t b(o,t) \leq 0, \quad t \in [0, T_{sing}),$$

by (2.10.2). We deduce

$$b(p_i, t_i) \to 0 \text{ as } i \to \infty.$$

Consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(p_i,t_i)} g\left(t_i + tb^2(p_i,t_i)\right), \quad t \in [-t_i b^{-2}(p_i,t_i), 0].$$

These satisfy the curvature bound

$$|Rm_{g_i(t)}|_{g_i(t)} \leq C^2 C_1 \text{ on } M_2 \times [-t_i b^{-2}(p_i,t_i), 0].$$

By Theorem 2.7.3 the rescaled metrics $g_i(t)$ are κ-non-collapsed at larger and larger scales. Hence by Corollary 2.8.2 the Ricci flows $(M_2, g_i(t), p_i)$ subsequentially converge to a pointed ancient Ricci flow $(M_2, g_\infty(t), p_\infty)$, $-\infty < t < 0$, with $g_\infty(t) \in \mathcal{I}$. By Theorem 2.11.1 it follows that $g_\infty(t)$ is stationary and homothetic to the Eguchi-Hanson metric. □

Proof of technical lemmas

In this subsection we collect the proofs of the technical lemmas we relied on above.

Proof of Lemma 2.11.6. We apply [**?**, Theorem 9.2] to prove this lemma. Define

$$r = Q^2.$$

Then

$$\begin{aligned} f' &= 2Qf_r \\ f'' &= 2f_r + 4rf_{rr}, \end{aligned}$$

where $'$ denotes the derivative with respect to Q and subscript r denotes the derivative with respect to r. Rewriting the differential equation (2.11.6) with respect to the independent variable r, we obtain

$$rf_{rr} = \frac{1}{2(1-r)}\left(6r - 4 - rf\right)f_r + \frac{f}{8\left(1-r\right)^2}\left(f^2 r + 3fr - 6f - 6r + 8\right) \qquad (2.11.11)$$

At $r = 0$ the right hand side must equal zero, which can be ensured by requiring

$$f_r(0) = \frac{1}{2}f_0 - \frac{3}{8}f_0^2$$

Now define

$$\begin{aligned} u_1 &= f - f_0 \\ u_2 &= f_r - f_r(0). \end{aligned}$$

Then (2.11.11) can be written as a system of equations of the form

$$r(u_i)_r = P_i(\vec{u}, r, f_0), \quad i = 1, 2,$$

where

$$\begin{aligned} P : \mathbb{R}^2 \times (-1,1) \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ (\vec{u}, r, f_0) &\longrightarrow P(\vec{u}, r, f_0) \end{aligned}$$

is an analytic vector-valued function of several variables satisfying

$$P(\vec{0}, 0, f_0) = 0$$

for all $f_0 \in (-1,1)$. A computation shows that

$$\frac{\partial P}{\partial u}\Big|_{(\vec{0},0,f_0)} = \begin{pmatrix} 0 & 0 \\ 1 - \frac{3}{2} f_0 & -2 \end{pmatrix}$$

This matrix has no positive integer eigenvalues and furthermore

$$B = \sup_{\substack{m \in \mathbb{N} \\ f_0 \in \mathbb{R}}} \left\| \left(m I_2 - \frac{\partial P}{\partial u}\Big|_{(\vec{0},0,f_0)} \right)^{-1} \right\| < \infty,$$

where I_2 is the 2×2 identity matrix. By [**?**, Theorem 9.2] the desired result follows. □

Proof of Lemma 2.11.7. At $Q = 0$, we have by L'Hôpital's Rule that

$$f'' = \frac{1}{4} f\,(4 - 3f) > 0 \ \text{ for } 0 < f(0) < \frac{4}{3}. \tag{2.11.12}$$

Furthermore, at an extremum of f we have

$$f'' = \frac{2f}{4\,(1 - Q^2)^2} \left(f^2 Q^2 + 3 f Q^2 - 6 f - 6 Q^2 + 8 \right). \tag{2.11.13}$$

Defining the polynomial

$$p_1(f, Q^2) = f^2 Q^2 + 3 f Q^2 - 6 f - 6 Q^2 + 8$$

we see that

$$\partial_{Q^2} p_1 = f^2 + 3f - 6 < 0 \ \text{ for } 0 < f \leq 1.$$

Therefore

$$p_1(f, Q) > p_1(f, 1) = f^2 - 3f + 2 \geq 0 \ \text{ for } 0 < f \leq 1 \text{ and } 0 \leq Q < 1.$$

From (2.11.13) it then follows that $f' > 0$ for as long as $0 < f \leq 1$. □

Proof of Lemma 2.11.8. We argue by contradiction. Assume there does not exist such a $Q_\theta < 1$. Then by Lemma 2.11.7 we have $f' > 0$ on $Q \in (0,1)$ and hence

$$\lim_{Q \to 1^-} f(Q) = l \leq 1$$

exists. By (2.11.6) we have

$$\begin{aligned} 4\left(1-Q^2\right)^2 f'' = & -4\left(1-Q^2\right)\left(Q^2 f - 5Q^2 + 3\right)\frac{f'}{Q} \\ & + 2f\left(Q^2\left(f^2+3f-6\right)+8-6f\right). \end{aligned}$$

For $Q^2 > 1 - \frac{\theta}{4}$ and $0 < f < 1$ we have

$$Q^2 f - 5Q^2 + 3 < 3 - 4Q^2 < -1 + \theta$$

and

$$Q^2\left(f^2+3f-6\right)+8-6f > (2-f)(1-f),$$

as for $0 < f < 1$

$$f^2 + 3f - 6 < 0.$$

Hence for $Q^2 > 1 - \frac{\theta}{4}$ and $0 < f < 1$ we obtain the following inequality

$$f'' \geq \alpha \frac{f'}{1-Q} + \beta \frac{1-f}{(1-Q)^2}, \tag{2.11.14}$$

where

$$\begin{aligned} \alpha &= \frac{1-\theta}{Q(1+Q)} \\ \beta &= \frac{f\,(2-f)}{2\,(1+Q)^2}. \end{aligned}$$

Furthermore we observe that

$$\begin{aligned} \alpha &\to \frac{1}{2} \ \text{ as } \ f, Q \to 1 \\ \beta &\to \frac{1}{8} \ \text{ as } \ f, Q \to 1. \end{aligned}$$

If $l < 1$, then there would exists a $Q_* < 1$ such that for $Q \geq Q_*$ we have

$$f'' \geq \frac{1}{10}\frac{l(1-l)(2-l)}{(1-Q)^2}.$$

Here $\frac{1}{10}$ can be replaced by any positive number less that $\frac{1}{8}$. However, integrating this differential inequality shows that in this case f would reach 1 before $Q = 1$, leading to a contradiction of our assumption. Therefore we may assume that $l = 1$.

Defining

$$g(Q) = 1 - f(Q)$$

we obtain the differential inequality

$$g''(Q) \leq \alpha \frac{g'(Q)}{1-Q} - \beta \frac{g(Q)}{(1-Q)^2}. \tag{2.11.15}$$

By Lemma 2.11.7 we know that $g(Q) > 0$ and $g'(Q) < 0$ on $Q \in (0, 1)$.

Claim 1: The function $g(Q)$ reaches zero before $Q = 1$.

Proof of Claim: By our assumption that $l = 1$ we know that there exists a $Q_* < 1$ such that for $Q > Q_*$

$$g(Q) < \theta.$$

Furthermore, by choosing $Q_* < 1$ sufficiently close to 1, we may assume that for $Q \geq Q_*$

$$g''(Q) \leq \frac{3}{7} \frac{g'(Q)}{1-Q} - \frac{5}{49} \frac{g(Q)}{(1-Q)^2}, \tag{2.11.16}$$

as $\frac{3}{7} < \frac{1}{2}$ and $\frac{5}{49} < \frac{1}{8}$. Now take the substitution

$$g(Q) = u(r)$$

for

$$r = -\ln(1-Q).$$

Then the (2.11.16) becomes

$$\frac{d^2 u}{dr^2} + \frac{4}{7} \frac{du}{dr} + \frac{5}{49} u \leq 0.$$

The corresponding ordinary differential equation is of oscillatory type, which motivates the substitution

$$u(r) = e^{-\frac{2}{7} r} v(r)$$

yielding the inequality

$$\frac{d^2 v}{dr^2} \leq -\frac{1}{49} v.$$

Hence v reaches 0 in finite r, which tracing back the substitutions, shows that g must reach zero before $Q = 1$. ■

Now it remains to prove the assertion (1), (2) and (3). We prove (1). First fix a $\theta \in (0, 1)$. By Lemma 2.11.7 we know that $f'_\theta(Q_\theta) > 0$. Now extend the solution f_θ of (2.11.6) to the interval $[0, Q_\theta + c]$, $c > 0$, such that $f'_\theta(Q) > 0$ on $(0, Q_\theta + c]$. By the continuous dependence of $f_\theta(Q)$ on θ it follows that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\theta - \theta'| \leq \delta$ implies $|Q_\theta - Q_{\theta'}| < \epsilon$. To prove the continuity of Q_θ at $\theta = 1$, note that $Q_1 = 1$, $f_1(0) = 1$ and

$f_1' = 0$. Then recall from the proof of Lemma 2.11.7 that $f''(0) > 0$ when $0 < f(0) < \frac{4}{3}$. Now applying the same argument as above yields continuity of Q_θ at $\theta = 1$ and therefore proves (1).

Assertion (2) follows from the fact that for the initial condition $f(0) = 0$ the corresponding solution to the ODE (2.11.6) is $f(Q) = 0$. By the continuous dependence of f on $f(0) = 0$ we deduce that

$$Q_\theta \to 1 \text{ as } \theta \to 0.$$

Finally, assertion (3) follows by definition. □

Proof of Lemma 2.11.11. First note that by Lemma 2.11.7 we know that $f, f' \geq 0$ on $[0, Q_\theta]$. Solving the ODE (2.11.6) for f'', we obtain

$$\begin{aligned} f'' = \frac{1}{2Q\left(1-Q^2\right)^2}\Big(&2Q^4 f f' - 10Q^4 f' - 2Q^2 f f' + 16Q^2 f' \\ &- 6f' + Q^3 f^3 + 3Q^3 f^2 - 6Q^3 f - 6Q f^2 + 8Q f\Big) \end{aligned} \tag{2.11.17}$$

Substituting expression (2.11.17) into (2.11.9) yields

$$\begin{aligned} A_2 = -\frac{1}{2\left(1-Q^2\right)^2}\Big(&2Q^3\left(1-Q^2\right)\left(2-Q^2 f\right) f' \\ &+ f^3Q^6 + 3f^2Q^6 - 6f^2Q^4 - 2fQ^6 + 4fQ^2 + 4Q^6 - 12Q^2 + 8\Big) \end{aligned}$$

Defining

$$p_2(f, Q^2) = f^3Q^6 + 3f^2Q^6 - 6f^2Q^4 - 2fQ^6 + 4fQ^2 + 4Q^6 - 12Q^2 + 8$$

we then only need to check that

$$p_2 \geq 0 \text{ for } 0 \leq f, Q \leq 1.$$

Defining

$$\tilde{p}_2(F, Q^2) = p_2(f, Q^2)$$

for

$$F = fQ^2$$

we see that

$$\partial_{Q^2}\tilde{p}_2\Big|_F = 3F^2 - 4FQ^2 + 12\left(Q^4 - 1\right) \leq 0 \text{ for } 0 \leq F, Q \leq 1$$

with equality only at $F = 0$, $Q = 1$. Therefore the minimum of p_2 is attained when $Q = 1$, in which case we have we have

$$\tilde{p}_2(F, 1) = (F-2)(F-1)F \geq 0 \text{ for } 0 \leq F \leq 1.$$

As $0 \leq Q \leq Q_\theta < 1$, we actually have $p_2(f, Q^2) > 0$ on $(f, Q) \in [0,1] \times [0, Q_\theta]$ and the result follows. □

Proof of Lemma 2.11.13. A computation shows that

$$\begin{aligned}\beta = &-\left(Q^2 f + 2Q^2 - 2\right)^2 f'' \\ &+ \left(-3Q^4 f^2 - 14Q^4 f + 12Q^2 f - 20Q^4 + 32Q^2 - 12\right)\frac{f'}{Q} \\ &- 4Q^2 f^2 - 12Q^2 f + 16f\end{aligned} \tag{2.11.18}$$

and

$$\begin{aligned}\gamma = &-\left(1-Q^2\right)^2 f'' + (1-Q^2)(2Q^2 f + 11Q^2 - 9)\frac{f'}{Q} \\ &+ 2Q^2 f^3 + 6Q^2 f^2 - 12f^2 - 6Q^2 f + \frac{30f}{Q^2} - 20f - 18Q^2 + \frac{54}{Q^2} - \frac{36}{Q^4},\end{aligned} \tag{2.11.19}$$

where we omitted the dependence of f on θ and Q for brevity. We first show that $\beta > 0$ in the region

$$R_1 = \left\{(f,Q)\,\middle|\, 0 \le Q \le Q_\theta, 0 < f \le \min\left(1, 3\frac{1-Q^2}{Q^2}\right)\right\}$$

of the f-Q-plane. Plugging the expression (2.11.17) of f'' into the expression (2.11.18) for β, we obtain

$$\begin{aligned}2\left(1-Q^2\right)^2\beta = &\, 2ff'Q(1-Q^2)\left(Q^4f^2 + 2Q^4 f - 4Q^2 f - 2Q^4 - 2Q^2 + 4\right) \\ &+ f^2\Big(-Q^6 f^3 - 7Q^6 f^2 + 10Q^4 f^2 - 10Q^6 f + 36Q^4 f \\ &\qquad - 28Q^2 f + 4Q^6 + 8Q^4 - 36Q^2 + 24\Big)\end{aligned}$$

An application of L'Hôpital's Rule shows that $\beta = 12f^2 > 0$ at $Q = 0$ and therefore we may assume that $Q > 0$. Recall that $f, f' > 0$ on $(0, Q_\theta)$ by Lemma 2.11.7. Hence it suffices to show that the polynomials

$$p_3(f, Q^2) = Q^4 f^2 + 2Q^4 f - 4Q^2 f - 2Q^4 - 2Q^2 + 4$$

and

$$\begin{aligned}p_4(f, Q^2) = &-Q^6 f^3 - 7Q^6 f^2 + 10Q^4 f^2 - 10Q^6 f + 36Q^4 f \\ &- 28Q^2 f + 4Q^6 + 8Q^4 - 36Q^2 + 24\end{aligned}$$

are positive on $R_1 \cap \{Q > 0\}$. A computation shows

$$\partial_{Q^2} p_3 = 2\left(f^2Q^2 - 1\right) + 4f\left(Q^2 - 1\right) - 4Q^2 \le 0$$

and hence for every $(f, Q) \in R_1$ we have

$$p_3(f, Q^2) \ge p_3\left(f, \frac{3}{3+f}\right) = \left(\frac{f}{3+f}\right)^2 > 0.$$

To show that $p_4 > 0$ on R_1 is more complicated. For this we introduce the variable

$$F = fQ^2$$

and polynomial

$$\tilde{p}_4(F, Q^2) = p_4(f, Q^2)$$

Then

$$\begin{aligned}\tilde{p}_4(F, Q^2) = 24 - 28F + 10F^2 - F^3 + \left(-7F^2 + 36F - 36\right) Q^2 \\ + (8 - 10F)Q^4 + 4Q^6\end{aligned}$$

which gives

$$\partial_{Q^2}\tilde{p}_4 = -7F^2 - 4F\left(-9 + 5Q^2\right) + 4\left(-9 + 4Q^2 + 3Q^4\right).$$

As this expression is concave in F one can easily check that in the region

$$0 < F \leq \min\left(Q^2, 3\left(1 - Q^2\right)\right), 0 \leq Q \leq 1$$

of the Q^2-F-plane we have

$$\partial_{Q^2}\tilde{p}_4 \leq 0$$

and thus

$$\tilde{p}_4(F, Q^2) \geq \tilde{p}_4\left(F, \frac{3-F}{3}\right) = \frac{1}{27}F\left(2F^2 - 3F + 18\right) > 0.$$

From this we conclude that $p_4 > 0$ on $R_1 \cap \{Q > 0\}$ and hence $\beta > 0$ on R_1.

We adopt the same procedure to show that $\gamma > 0$ in the region

$$R_2 = \left\{(f, Q) \,\middle|\, 0 \leq Q \leq Q_\theta, 3\frac{1-Q^2}{Q^2} \leq f \leq 1\right\}.$$

Substituting the expression (2.11.17) for f'' into the expression (2.11.19) for γ we obtain

$$\begin{aligned}\gamma = 3\left(1 - Q^2\right)\left(Q^2 f + 2Q^2 - 2\right)\frac{f'}{Q} + \frac{1}{Q^4}\Big(\frac{3f^3Q^6}{2} + \frac{9f^2Q^6}{2} - 9f^2Q^4 \\ - 3fQ^6 - 24fQ^4 + 30fQ^2 - 18Q^6 + 54Q^2 - 36\Big)\end{aligned}$$

First notice that for any point $(f, Q) \in R_2$

$$3\frac{1-Q^2}{Q^2} \leq 1$$

and hence

$$\sqrt{\frac{3}{4}} \leq Q \leq Q_\theta < 1.$$

Then from

$$f \geq 3\frac{1-Q^2}{Q^2}$$

it follows that

$$Q^2 f + 2Q^2 - 2 \geq 1 - Q^2 > 0 \ \text{ on } \ R_2.$$

Therefore the first term in the expression for γ is positive and we only need to prove non-negativity of the second term. For this define the polynomials

$$\begin{aligned} p_5(f, Q^2) = & \frac{3f^3Q^6}{2} + \frac{9f^2Q^6}{2} - 9f^2Q^4 - 3fQ^6 - 24fQ^4 \\ & + 30fQ^2 - 18Q^6 + 54Q^2 - 36 \end{aligned}$$

and

$$\begin{aligned} \tilde{p_5}(F, Q^2) = & \frac{3F^3}{2} + \frac{9F^2Q^2}{2} - 9F^2 - 3FQ^4 - 24FQ^2 \\ & + 30F - 18Q^6 + 54Q^2 - 36, \end{aligned}$$

where we again took $F = fQ^2$. Computing the partial derivatives

$$\begin{aligned} \partial_{Q^2}\tilde{p}_5 &= \frac{9F^2}{2} - 6FQ^2 - 24F - 54Q^4 + 54 \\ \partial_F \tilde{p}_5 &= \frac{9F^2}{2} + 9FQ^2 - 18F - 3Q^4 - 24Q^2 + 30 \end{aligned}$$

We deduce that at an local extrema $\partial_{Q^2}\tilde{p}_3 = \partial_F \tilde{p}_4 = 0$

$$F = \frac{-17Q^4 + 8Q^2 + 8}{5Q^2 + 2}$$

and

$$80 + 144Q^2 - 188Q^4 - 200Q^6 + 307Q^8 = 0.$$

In Lemma 2.11.15 below we show that the equation for Q^2 has no zeros in the interval $Q^2 \in [\frac{3}{4}, 1]$. Therefore $p_5(F, Q^2)$ has no local extrema in the region R_3 of the (F, Q^2)-plane enclosed by the curves

$$\begin{aligned} L_1 &: Q^2 = 1, 0 \leq F \leq 1 \\ L_2 &: \frac{2}{3} \leq Q^2 \leq 1, 0 \leq F \leq 1 \\ L_3 &: \frac{2}{3} \leq Q^2 \leq 1, F = 3\left(1 - Q^2\right) \end{aligned}$$

As the set of the (F, Q^2)-plane corresponding to R_2 is a subset of R_3 it suffices to check non-negativity of $\tilde{p}_5$ on the boundary of the region R_3. There we have

$$\tilde{p}_5(F, 1) = \frac{3}{2}F\left(1 - F\right)\left(2 - F\right) \geq 0 \quad \text{ on } L_1$$

and

$$\tilde{p}_5(1, Q^2) = \frac{3}{2}(1 - Q^2)\left(12Q^4 + 14Q^2 - 9\right) \geq 0 \quad \text{on } L_2$$

and

$$\tilde{p}_5\left(3(1 - Q^2), Q^2\right) = \frac{9}{2}\left(1 - Q^2\right)\left(2Q^4 - 3Q^2 + 3\right) \geq 0 \quad \text{on } L_3$$

This concludes the proof. □

Lemma 2.11.15. *The equation*

$$80 + 144r - 188r^2 - 200r^3 + 307r^4 = 0$$

has no roots in the interval $[0, 1]$.

Proof. For $r \in [0, 1]$ we have

$$\begin{aligned} 80 + 144r - 188r^2 - 200r^3 + 307r^4 &\geq (80 - 6r) + 150r - 200r^2 - 200r^3 + 300r^4 \\ &\geq 24 + 50\left(1 + 3r - 4r^2 - 4r^3 + 6r^4\right). \end{aligned}$$

Then we see that

$$\begin{aligned} 1 + 3r - 4r^2 - 4r^3 + 6r^4 &= \left(1 - 2r^2\right)^2 + r\left(2r^3 - 4r^2 + 3\right) \\ &\geq \left(1 - 2r^2\right)^2 + r\left(2r^4 - 4r^2 + 3\right) \\ &\geq \left(1 - 2r^2\right)^2 + r\left(2(r^2 - 1)^2 + 1\right) \\ &\geq 0 \end{aligned}$$

This concludes the proof. □

2.12 Discussion of blow-up limits in $k = 2$ case

In this section we investigate the possible blow-up limits of a Ricci flow $(M_2, g(t))$, $t \in [0, T_{sing})$, starting from an initial metric $g(0) \in \mathcal{I}$ with $\sup_{p \in M_2} b(p, 0) < \infty$. By Lemma 2.10.5 and Corollary 2.11.2 we know that such flows develop a Type II singularity modeled on the Eguchi-Hanson space as the area of the non-principal orbit S^2_o shrinks to zero. One expects, however, that at larger distance scales from S^2_o one could also see other blow-up limits. The goal of this section is to show that these are in fact limited to the following four possibilities: (i) the Eguchi-Hanson space, (ii) the flat $\mathbb{R}^4/\mathbb{Z}_2$ orbifold, (iii) the 4d Bryant soliton quotiented by $\mathbb{Z}_2$ and (iv) the shrinking cylinder $\mathbb{R} \times \mathbb{R}P^3$.

Next, we state the main theorem of this section:

Theorem 2.12.1 (Blow-up limits)**.** *Let $(M_2, g(t))$, $t \in [0, T_{sing})$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ (see Definition 2.7.2) with $\sup_{p \in M_2} b(p, 0) < \infty$. Let (p_i, t_i) be a sequence of points in spacetime with $b(p_i, t_i) \to 0$. Passing to a subsequence, we may assume that we are in one of the four cases listed below.*

(i) $\lim_{i\to\infty} \frac{b(p_i,t_i)}{b(o,t_i)} < \infty$

(ii) $\lim_{i\to\infty} \frac{b(p_i,t_i)}{b(o,t_i)} = \infty$ *and* $\lim_{i\to\infty} b_s(p_i,t_i) = 1$

(iii) $\lim_{i\to\infty} \frac{b(p_i,t_i)}{b(o,t_i)} = \infty$ *and* $\lim_{i\to\infty} b_s(p_i,t_i) \in (0,1)$

(iv) $\lim_{i\to\infty} \frac{b(p_i,t_i)}{b(o,t_i)} = \infty$ *and* $\lim_{i\to\infty} b_s(p_i,t_i) = 0$

Consider the dilated Ricci flows

$$g_i(t) = \frac{1}{b^2(p_i,t_i)} g\left(t_i + b^2(p_i,t_i)t\right), \quad t \in [-b(p_i,t_i)^{-2}t_i, 0].$$

Then $(M_2, g_i(t), p_i)$*,* $t \in [-b(p_i,t_i)^{-2}t_i, 0]$*, subsequentially converges, in the Cheeger-Gromov sense, to an ancient Ricci flow* $(M_\infty, g_\infty(t), p_\infty)$*,* $t \in (-\infty, 0]$*. Depending on the limiting property of the sequence* (p_i,t_i) *we have:*

(i) $M_\infty \cong M_2$ *and* $g_\infty(t)$ *is stationary and homothetic to the Eguchi-Hanson metric*

(ii) $M_\infty \cong \mathbb{R}^4 \setminus \{0\}/\mathbb{Z}_2$ *and* $g_\infty(t)$ *can be extended to a smooth orbifold Ricci flow on* $\mathbb{R}^4/\mathbb{Z}_2$ *that is stationary and isometric to the flat orbifold* $\mathbb{R}^4/\mathbb{Z}_2$

(iii) $M_\infty \cong \mathbb{R}^4 \setminus \{0\}/\mathbb{Z}_2$ *and* $g_\infty(t)$ *can be extended to a smooth orbifold Ricci flow on* $\mathbb{R}^4/\mathbb{Z}_2$ *that is homothetic to the 4d Bryant soliton quotiented by* $\mathbb{Z}_2$

(iv) $M_\infty \cong \mathbb{R} \times \mathbb{R}P^3$ *and* $g_\infty(t)$ *is homothetic to a shrinking cylinder*

Remark 2.12.2. Notice that in Theorem 2.12.1 we do not claim that all blow-up limits (i)-(iv) actually occur. If the Eguchi-Hanson singularity is isolated one would only see (i) and (ii).

Outline of proof

Assume we are given a sequence of points (p_i,t_i) in spacetime with $b(p_i,t_i) \to 0$. Consider the rescaled metrics

$$g_i(t) = \frac{1}{b(p_i,t_i)^2} g(t_i + b(p_i,t_i)^2 t), \quad t \in [-b(p_i,t_i)^{-2}t_i, 0],$$

normalized such that $b(p_i,0) = 1$. By passing to a subsequence we may assume that either

$$\text{(I)} \ \sup_i \frac{b(p_i,t_i)}{b(o,t_i)} < \infty \quad \text{or} \quad \text{(II)} \ \lim_{i\to\infty} \frac{b(p_i,t_i)}{b(o,t_i)} = \infty.$$

In case (I) we know by Corollary 2.11.2 that $(M_2, g_i(t), p_i)$ subsequentially converges to the Eguchi-Hanson space, which is the blow-up limit (i) from above. Therefore we only need to investigate the behavior in case (II), i.e. at scales larger than the forming Eguchi-Hanson singularity. At these scales Lemma 2.12.3 yields very important geometric information. In particular, we show that for every $\epsilon > 0$ there exist constants $C, \delta > 0$ such that the following holds: For all points (p,t) in spacetime at which $Cb(o,t) \leq b(p,t) \leq \delta$ we have

- $Q \geq 1 - \epsilon$
- $T_{F_1} \coloneqq bb_{ss} + 1 - b_s^2 \geq -\epsilon$
- $T_{F_2} \coloneqq bb_{ss} + 1 - b_s^2 - (1 - b_s^2)^2 \leq \epsilon$
- $\partial_t b^2 \leq \epsilon$

Hence a blow-up limit $(M_\infty, g_\infty(t), p_\infty)$ in case (II) satisfies $Q = 1$, $T_{F_1} \geq 0$ and $T_{F_2} \leq 0$. Therefore M_∞ is rotationally symmetric and satisfies

$$\frac{1 - b_s^2}{b^2} - \frac{(1 - b_s^2)^2}{b^2} \leq -\frac{b_{ss}}{b} \leq \frac{1 - b_s^2}{b^2}.$$

As $-\frac{b_{ss}}{b}$ and $\frac{1-b_s^2}{b^2}$ are the only non-zero components of the curvature tensor of a rotationally symmetric metric, we see that blow-up limits of case (II) satisfy the curvature bound

$$|Rm_{g_\infty(t)}|_{g_\infty(t)} \leq c\frac{1 - b_s^2}{b^2}$$

for some universal constant $c > 0$.

We now briefly explain some of the geometric intuition behind the quantities T_{F_1} and T_{F_2} for rotationally symmetric metrics. When $T_{F_1} = 0$ the underlying space is of constant curvature and therefore isometric to a sphere, the flat plane or hyperbolic space, depending on the sign of the scalar curvature. On the other hand solving the ODE $T_{F_2} = 0$ one can show that $b_s \to 0$ as $s \to \infty$ and the underlying space is asymptotically cylindrical. Thus blow-up limits in case (II) are rotationally symmetric spaces that are 'sandwiched' between a sphere and an asymptotically cylindrical space.

We need to divide case (II) into three subcases in order to investigate the possible blow-up limits: By passing to a subsequence we may assume that

$$\text{(II.a) } b_s(p_i, t_i) \to 1 \ \text{ or } \ \text{(II.b) } b_s(p_i, t_i) \to \eta \in (0,1) \ \text{ or } \ \text{(II.c) } b_s(p_i, t_i) \to 0.$$

For (II.a) and (II.c) we show in Lemma 2.12.9 and Lemma 2.12.6 that $(M_2, g_i(t), p_i)$ subsequentially converges to the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$ and the shrinking cylinder $\mathbb{R} \times \mathbb{R}P^3$, respectively. The main idea is that by the strong maximum principle applied to the evolution equation (2.13.4) of b_s when $Q = 1$ a minimum $b_s = 0$ or a maximum $b_s = 1$ can only be attained if b_s is constant everywhere.

Proving that the blow-up limit in case (II.b) is an ancient orbifold Ricci flow, which is homothetic to the 4d Bryant soliton quotiented by $\mathbb{Z}_2$, is trickier. The construction is carried out in Lemma 2.12.8, the proof of which we sketch here: Fix a $T > 0$ and define

$$E_{p,t,n} \coloneqq \left\{ p' \in M_2 \,\middle|\, b(p', t) > \frac{b(p,t)}{n} \right\} \subseteq M_2.$$

Then consider the rescaled metrics $g_i(t)$, defined as above, on the parabolic neighborhoods

$$\Omega_{i,n} \coloneqq E_{p_i,t_i,n} \times [-T-1, 0]$$

in spacetime. By Lemma 2.12.3 we know that $\partial_t b^2 \to 0$ uniformly as $b \to 0$. Hence from the curvature bound

$$|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{b^2} \tag{2.12.1}$$

of Theorem 2.7.5, we see that the curvature of $g_i(t)$ is bounded by Cn^2 on $\Omega_{i,n}$ for i sufficiently large and $C > 0$ some constant. The difficulty in constructing the limiting orbifold flow arises from the fact that the curvature bound (2.12.1) degenerates as $n \to \infty$. We get around this by exploiting the inequalities on T_{F_1} and T_{F_2} derived in Lemma 2.12.3, to find a uniform curvature bound independent of n. From here it is then easy to construct the orbifold Ricci flow $g_\infty(t)$, $t \in [-T, 0]$, on $\mathbb{R}^4 \setminus \{0\}/\mathbb{Z}_2$ by taking the limit $n \to \infty$. Via Lemma 2.14.2, and Theorem 2.14.1 in the Appendix B, we show that $g_\infty(t)$ can be extended to a smooth orbifold Ricci flow on $\mathbb{R}^4/\mathbb{Z}_2$. Apriori the curvature bound of $g_\infty(t)$ on $\mathbb{R}^4/\mathbb{Z}_2 \times [0, T]$ may deteriorate as $T \to \infty$. Nevertheless we can use a diagonal argument to construct an ancient orbifold Ricci flow on $\mathbb{R}^4/\mathbb{Z}_2$. Hamilton's trace Harnack inequality then implies that $g_\infty(t)$ has bounded curvature on $\mathbb{R}^4/\mathbb{Z}^2 \times (-\infty, 0]$. Finally, we apply the result [LZ18] by Xiaolong Li and Yongjia Zhang to deduce that $g_\infty(t)$ is homothetic to the 4d Bryant soliton quotiented by $\mathbb{Z}_2$.

Proof of main theorem

We begin by proving the central lemma of this section, which yields important geometric information on the high curvature regions of a Ricci flow $(M_2, g(t))$ as in the Theorem 2.12.1.

Lemma 2.12.3. *Let $(M_2, g(t))$, $t \in [0, T_{sing})$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ with $\sup_{p \in M_2} b(p, 0) < \infty$. Then for every $\epsilon \in (0, 1)$ there exist constants $C, \delta > 0$ such that at all points (p, t) in spacetime with $Cb(o, t) \leq b(p, t) \leq \delta$ the following inequalities hold:*

(i) $Q \geq 1 - \epsilon$

(ii) $bb_{ss} \leq \epsilon$

(iii) $T_{F_1} \coloneqq bb_{ss} + 1 - b_s^2 \geq -\epsilon$

(iv) $T_{F_2} \coloneqq bb_{ss} + 1 - b_s^2 - (1 - b_s^2)^2 \leq \epsilon$

(v) $\partial_t b^2 \leq \epsilon$

Remark 2.12.4. Inequality (ii) is implied by (iv) for metrics in $\mathcal{I}$. However, we need (ii) as an intermediate result before proving (iv).

Proof. Fix $\epsilon \in (0,1)$. Recall the following facts of the Eguchi-Hanson space (M_2, g^E) derived in section 2.4:

(a) $x = y = 0$ on M_2

(b) $Q \to 1$ as $s \to \infty$

(c) g^E is normalized such that its warping function b^E satisfies $b^E(0) = 1$.

Using (a) a computation shows that

$$\begin{aligned} bb_{ss} &= 2\left(1 - Q^2\right) \\ T_{F_1} &= 3\left(1 - Q^2\right) \\ T_{F_2} &= 3\left(1 - Q^2\right) - \left(1 - Q^2\right)^2 \\ \partial_t b^2 &= 0 \end{aligned}$$

on (M_2, g^E). Pick $C > 10$ such that on (M_2, g^E) we have $Q > 1 - \epsilon$, $bb_{ss} < \epsilon$, $T_{F_1} > -\epsilon$ and $T_{F_2} < \epsilon$ whenever $s > C$. This is possible by property (b).

Take a path $\gamma : [0, T_{sing}) \to M_2$ such that

$$s(\gamma(t), t) = Cb(o, t),$$

where we recall that $s(p, t)$ is the distance of a point $p \in M_2$ from the non-principal orbit S^2_o at the tip of M_2. By Corollary 2.11.2 we know that at distance scales comparable to $b(o, t)$ away from S^2_o we converge to the Eguchi-Hanson space as $t \to T_{sing}$. From the scale-invariance of Q, bb_{ss}, T_{F_1}, T_{F_2} and $\partial_t b^2$ it follows that at spacetime points $(\gamma(t), t)$ inequalities (i)-(v) eventually hold as $t \to T_{sing}$.

Let A be the set of all sequences of points $\{(p_i, t_i)\}_{i \in \mathbb{N}}$ in spacetime satisfying the following two properties:

1. $b(p_i, t_i) \geq Cb(o, t_i)$
2. $b(p_i, t_i) \to 0$ as $i \to \infty$

Note that property (2) implies that for such sequences $t_i \to T_{sing}$ as $i \to \infty$.

We first prove inequality (i), arguing by contradiction. Assume that

$$\iota := \inf\left\{ \liminf_{i \to \infty} Q(p_i, t_i) \,\middle|\, \{(p_i, t_i)\}_{i \in \mathbb{N}} \in A \right\} < 1 - \epsilon.$$

Then there exists a sequence $\{A_n\}_{n \in \mathbb{N}}$ of sequences $A_n = \{(p_{n,i}, t_{n,i})\}_{i \in \mathbb{N}}$ of points in spacetime satisfying properties (1) and (2) above, and

$$\lim_{n \to \infty} \liminf_{i \to \infty} Q(p_{n,i}, t_{n,i}) = \iota.$$

For each $n \in \mathbb{N}$ take $N(n) \in \mathbb{N}$ such that for $m \geq N(n)$ we have

$$\left| \liminf_{i\to\infty} Q(p_{m,i}, t_{m,i}) - \iota \right| \leq \frac{1}{n}.$$

For each $n \in \mathbb{N}$ take $I(n) \in \mathbb{N}$ such that

$$\left| Q(p_{n,I(n)}, t_{n,I(n)}) - \liminf_{i\to\infty} Q(p_{n,i}, t_{n,i}) \right| \leq \frac{1}{n}$$

and

$$b(p_{n,I(n)}, t_{n,I(n)}) \leq \frac{1}{n}.$$

Let $(p_n, t_n) = (p_{N(n),I(N(n))}, t_{N(n),I(N(n))})$ for $n \in \mathbb{N}$. Then we see that

$$Q(p_i, t_i) \to \iota \text{ as } i \to \infty$$

and both properties (1) and (2) from above hold.

Before we carry on recall that by Theorem 2.7.5 there exists a $C_1 > 0$ such that

$$|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{b^2} \text{ on } M_2 \times [0, T_{sing}).$$

Recall also Theorem 2.7.3, from which it follows that there exist constants $\kappa, \rho > 0$ such that $g(t)$ is κ-non-collapsed at scale ρ. Next, consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(p_i, t_i)} g(t_i + tb^2(p_i, t_i)), \quad [-\Delta t, 0],$$

where $\Delta t > 0$ is chosen such that Proposition 2.8.3 holds. Then $(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i)$ subsequentially converges to a pointed Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$, $t \in [-\Delta t, 0]$. By construction

$$b(p_\infty, 0) = 1$$

and

$$Q(p_\infty, 0) = \iota < 1 - \epsilon.$$

Claim 1: $\lim_{i\to\infty} \frac{b(p_i,t_i)}{b(o,t_i)} = \infty$

Proof of Claim: We argue by contradiction. Assume there exists a $C' > 0$ such that after passing to a subsequence of (p_i, t_i)

$$\frac{b(p_i, t_i)}{b(o, t_i)} < C'.$$

Consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(p_i, t_i)} g(t_i + tb^2(p_i, t_i)), \quad [-b(p_i, t_i)^{-2} t_i, 0].$$

By Corollary 2.11.2 we see that $(M_2, g_i(t), p_i)$ subsequentially converges to $(M_2, g_\infty(t), p_\infty)$, where $g_\infty(t)$ is stationary and homothetic to the Eguchi-Hanson metric. By construction

$$1 = b(p_\infty, 0) \geq Cb(o, 0)$$

and

$$Q(p_\infty, 0) = \iota < 1 - \epsilon.$$

Furthermore,

$$s(p_\infty, 0) \geq b(p_\infty, 0) \geq Cb(o, 0),$$

where the first inequality follows from the fact that $1 \geq Q \geq b_s \geq 0$ for metrics in $\mathcal{I}$ and the second inequality follows from the definition of C. Thus

$$Q(p_\infty, 0) > 1 - \epsilon,$$

which is a contradiction and hence proves the claim. ■

Claim 2: For every $N \in \mathbb{N}$ eventually $\frac{b(p,t)}{b(o,t)} > N$ everywhere in $(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i) \times [-\Delta t, 0]$

Proof of Claim: Fix $N \in \mathbb{N}$. We argue by contradiction. After passing to a subsequence of (p_i, t_i), we may assume that there exists a sequence of spacetime points $(p_i', t_i') \in (C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i) \times [-\Delta t, 0]$ for which $\frac{b(p_i', t_i')}{b(o, t_i')} \leq N$. Consider the rescaled metrics

$$g_i'(t) = \frac{1}{b^2(p_i', t_i')} g(t_i' + tb^2(p_i', t_i')), \quad t \in [-b(p_i', t_i')^{-2} t_i', 0].$$

By Corollary 2.11.2, $(M_2, g_i'(t), p_i')$, $t \in [-b(p_i', t_i')^{-2} t_i', 0]$, converges to an ancient Ricci flow $(M_\infty, g_\infty'(t), p_\infty')$, $t \in (-\infty, 0]$, which is stationary and homothetic to the Eguchi-Hanson space. Note that on the non-principal orbit S_o^2

$$0 \geq \partial_t b^2(o, t) = 4by_s \geq -4bQ_s = -4k,$$

as $0 \geq y = b_s - Q \geq -Q$ and $y = Q = 0$ at o for metrics in $\mathcal{I}$. Hence for $\tau \in (0, \frac{1}{4k})$ the warping function b_i of the metric $g_i(t)$ satisfies

$$b_i(o, t) \geq 1 - 4k\tau > 0 \text{ for } t \in [-b(p_i', t_i')^{-2} t_i', \tau].$$

We deduce by Theorem 2.7.5 that $g_i(t)$ has bounded curvature on $M_2 \times [-b(p_i', t_i')^{-2} t_i', \tau]$. Hence $(M_2, g_i'(t), p_i')$, $t \in [-b(p_i', t_i')^{-2} t_i', \tau]$, also converges to the stationary Eguchi-Hanson space. In fact, inductively we can then show that for any $\tau > 0$ we converge to the Eguchi-Hanson space. As (p_i', t_i') converges to a point (p_∞', t_∞') in $\mathcal{C}_\infty \times [-\Delta t, 0]$, this implies that $\mathcal{C}_\infty \times [-\Delta t, 0]$ is a subset of a spacetime corresponding to the Eguchi-Hanson space, and therefore $\lim_{i \to \infty} \frac{b(p_i, t_i)}{b(o, t_i)} < \infty$. This, however, contradicts Claim 1. ■

Claim 3: $Q(p_\infty, 0) = \inf_{\mathcal{C}_\infty \times [-\Delta t, 0]} Q$

Proof of Claim: We argue by contradiction. If $Q(p', t') < \iota$ at a point $(p', t') \in \mathcal{C}_\infty \times [-\Delta t, 0]$, one could pick a sequence of points $(p'_i, t'_i) \in C_{g_i(0)}\left(p_i, \frac{1}{2}\right) \times [-\Delta t, 0]$ with $(p'_i, t'_i) \to (p', t')$ as $i \to \infty$. Then shifting back to the time of the Ricci flow $(M_2, g(t))$ via $T'_i = t_i + t'_i b(p_i, t_i)^2$ we see that the sequence $(p'_i, T'_i) \in M_2 \times [0, T_{sing})$ satisfies properties (1) and (2). The former property holds because of Claim 2. This, however, would contradict the definition of ι. ■

By property (1) of the sequence (p_i, t_i) we see that p_∞ does not lie on a non-principal orbit of $\mathcal{C}_\infty$ and therefore $Q(p_\infty, 0) = a(p_\infty, 0) > 0$. However, the evolution equation (2.5.1) of Q shows that the only attainable minima are 0 and 1, yielding a contradiction. This concludes the proof of (i).

We prove (ii)-(v) by the same strategy. Below we first prove inequality (ii) by contradiction. Assume that

$$\iota := \sup \left\{ \limsup_{i\to\infty} bb_{ss}\big|_{(p_i,t_i)} \;\middle|\; \{(p_i,t_i)\}_{i\in\mathbb{N}} \in A \right\} > \epsilon.$$

As before we can construct a sequence of points (p_i, t_i) in spacetime satisfying properties (1) and (2), and such that

$$\lim_{i\to\infty} bb_{ss}\big|_{(p_i,t_i)} = \iota.$$

Consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(p_i,t_i)} g(t_i + tb^2(p_i,t_i)), \quad [-\Delta t, 0],$$

where $\Delta t > 0$ is chosen such that Proposition 2.8.3 holds. Then $\left(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i\right)$ subsequentially converges to a pointed Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$. By construction

$$bb_{ss}\big|_{(p_\infty,0)} = \iota > \epsilon.$$

Furthermore, by the same arguments as in Claim 1 & 2 & 3, we have

$$bb_{ss}\big|_{(p_\infty,0)} = \sup_{\mathcal{C}_\infty\times[-\Delta t,0]} bb_{ss}$$

and hence

$$\partial_t bb_{ss}\Big|_{(p_\infty,0)} \geq 0.$$

By statement (i) of this lemma we know that $Q = 1$ on $\mathcal{C}_\infty \times [-\Delta t, 0]$. For $Q = 1$ the evolution equation for bb_{ss} is

$$\partial_t(bb_{ss}) = (bb_{ss})_{ss} - \frac{b_s}{b}(bb_{ss})_s - 4\frac{b_s^2}{b^2}\left(1 - b_s^2\right) - 2\frac{b_{ss}}{b}\left(bb_{ss} + 2b_s^2\right).$$

The derivation is carried out in the Appendix A, in the subsection on the evolution equations when $Q = 1$. From this we see that at the point $(p_\infty, 0)$ in spacetime we have

$$\partial_t(bb_{ss})\Big|_{(p_\infty,0)} < 0,$$

which is a contradiction. This proves (ii).

We prove inequality (iii) similarly. Assume that

$$\iota := \inf\left\{\liminf_{i\to\infty} T_{F_1}(p_i,t_i) \;\Big|\; \{(p_i,t_i)\}_{i\in\mathbb{N}} \in A\right\} < -\epsilon.$$

Pick $\{(p_i,t_i)\}_{i\in\mathbb{N}} \in A$ such that

$$\lim_{i\to\infty} T_{F_1}(p_i,t_i) = \iota.$$

As before, $\left(C_{g_i(0)}\left(p_i,\frac{1}{2}\right), g_i(t), p_i\right)$ subsequentially converges to a pointed Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$. By construction

$$T_{F_1}(p_\infty,0) = \inf_{\mathcal{C}_\infty\times[-\Delta t,0]} T_{F_1} = \iota < -\epsilon$$

and hence

$$\partial_t T_{F_1}\big|_{(p_\infty,0)} \le 0.$$

By inequality (i) of this lemma $Q = 1$ on $\mathcal{C}_\infty \times [-\Delta t, 0]$. For $Q = 1$ the evolution equation of T_{F_1} can be written as

$$\partial_t T_{F_1} = (T_{F_1})_{ss} - \frac{b_s}{b}(T_{F_1})_s - 8\frac{b_s^2}{b^2}T_{F_1}.$$

The derivation is carried out in the Appendix A. From this we see that at the point $(p_\infty, 0)$ in spacetime we have

$$\partial_t T_{F_1}\Big|_{(p_\infty,0)} \ge 0,$$

with equality if, and only if, $b_s\big|_{(p_\infty,0)} = 0$. Therefore we conclude that $b_s = 0$ at $(p_\infty, 0)$. Applying the strong maximum principle to the evolution equation (2.13.4) of b_s when $Q = 1$, it then follows that $b_s = 0$, and hence $b_{ss} = 0$, everywhere in $\mathcal{C}_\infty \times [-\Delta t, 0]$. This, however, implies $T_{F_1} = 1$ at $(p_\infty, 0)$, which is a contradiction and thus proves (ii).

We proceed to prove (iv) in the same fashion. Assume that

$$\iota := \sup\left\{\limsup_{i\to\infty} T_{F_2}(p_i,t_i) \;\Big|\; \{(p_i,t_i)\}_{i\in\mathbb{N}} \in A\right\} > \epsilon.$$

Pick $\{(p_i,t_i)\}_{i\in\mathbb{N}} \in A$ such that

$$\lim_{i\to\infty} T_{F_2}(p_i,t_i) = \iota.$$

As before, $\left(C_{g_i(0)}\left(p_i,\frac{1}{2}\right), g_i(t), p_i\right)$ subsequentially converges to a pointed Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$. By construction

$$T_{F_2}(p_\infty,0) = \sup_{\mathcal{C}_\infty\times[-\Delta t,0]} T_{F_2} = \iota > \epsilon.$$

Therefore

$$\partial_t T_{F_2}\Big|_{(p_\infty,0)} \geq 0.$$

By statement (i) and (ii) of this lemma we have $Q = 1$ and $bb_{ss} \leq 0$ on $\mathcal{C}_\infty \times [-\Delta t, 0]$. When $Q = 1$ the evolution equation of T_{F_2} can be written as

$$\partial_t T_{F_2} = (T_{F_2})_{ss} - \frac{b_s}{b}(T_{F_2})_s + \frac{1}{b^2}C_{F_2},$$

where C_{F_2} is a polynomial expression in bb_{ss} and $1 - b_s^2$. The derivation is carried out in the Appendix A. By Lemma 2.13.1 in the Appendix A, $C_{F_2} < 0$ whenever $T_{F_2} > 0$ and $bb_{ss} \leq 0$. This, however, implies

$$\partial_t T_{F_2}\Big|_{(p_\infty,0)} < 0,$$

which is a contradiction and thus proves (iv).

Finally we prove (v), also by contradiction. Assume that

$$\iota := \sup\left\{ \limsup_{i\to\infty} \partial_t b^2(p_i, t_i) \;\Big|\; \{(p_i, t_i)\}_{i\in\mathbb{N}} \in A \right\} > \epsilon.$$

Pick $\{(p_i, t_i)\}_{i\in\mathbb{N}} \in A$ such that

$$\lim_{i\to\infty} \partial_t b^2(p_i, t_i) = \iota.$$

As before, $\left(C_{g_i(0)}\left(p_i, \frac{1}{2}\right), g_i(t), p_i\right)$ subsequentially converges to a pointed Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$. By construction

$$\partial_t b^2\Big|_{(p_\infty,0)} = \iota > \epsilon,$$

as $\partial_t b^2$ is a scale-invariant quantity. By (i) we have $Q = 1$ on $\mathcal{C}_\infty \times [-\Delta t, 0]$ and the evolution equation (2.2.12) of b simplifies to

$$\partial_t b^2 = 2bb_{ss} + 4\left(b_s^2 - 1\right).$$

By inequality (iii) of this lemma we have

$$bb_{ss} \leq b_s^2 - 1 + \left(1 - b_s^2\right) \leq 0 \;\text{ on }\; \mathcal{C}_\infty \times [-\Delta t, 0],$$

as $b_s \in [0,1]$ for metrics in $\mathcal{I}$. This, however, implies that

$$\partial_t b^2 \leq 0,$$

which is a contradiction and thus proves (v). □

Lemma 2.12.5. *Let $(M_2, g(t))$, $t \in [0, T_{sing})$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ with $\sup_{p\in M_2} b(p,0) < \infty$. Then for every $\epsilon \in (0,1)$ there exists a $\delta > 0$ such that at all points (p,t) in spacetime at which $b(p,t) \leq \delta$ we have*

$$\partial_t b^2 \leq \epsilon.$$

Proof. Fix $\epsilon > 0$. By Lemma 2.12.3 we only need to prove that there exists a $\delta > 0$ such that the result holds when $b(p,t) \leq Cb(o,t) \leq \delta$, where $C > 0$ is as in Lemma 2.12.3. Note that

$$\partial_t b^2 = 0$$

on the Eguchi-Hanson space background. By Corollary 2.11.2 we know that on the scale $b(p,t) \leq Cb(o,t)$ we converge to the Eguchi-Hanson space as $t \to T_{sing}$. As $b(o,t) \to 0$ as $t \to T_{sing}$ we see that there exists a $\delta > 0$ such that

$$\partial_t b^2 \leq \epsilon$$

at all points (p,t) in spacetime at which $b(p,t) \leq Cb(o,t) \leq \delta$. This completes the proof. □

Below we prove the simplest case of Theorem 2.12.1.

Lemma 2.12.6. *Let $(M_2, g(t))$, $t \in [0, T_{sing})$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ with $\sup_{p \in M_2} b(p,0) < \infty$. Let (p_i, t_i) be a sequence of points in spacetime satisfying*

1. $b(p_i, t_i) \to 0$
2. $b_s(p_i, t_i) \to 0$
3. $b(p_i, t_i) > 2b(o, t_i)$

Consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(p_i,t_i)} g(t_i + b^2(p_i,t_i)t), \quad t \in [-b(p_i,t_i)^{-2}t_i, 0].$$

Then $(M_2, g_i(t), p_i)$, $t \in [-b(p_i,t_i)^{-2}t_i, 0]$, subsequentially converges, in the Cheeger-Gromov sense, to the shrinking cylinder $\mathbb{R} \times \mathbb{R}P^3$.

Proof. Fix $T > 0$. By Lemma 2.12.5, the curvature bound of Theorem 2.7.5 and the fact that $b_s \in [0,1]$ for metrics in $\mathcal{I}$, we see that the curvatures of $g_i(t)$ on the parabolic neighborhoods $C_{g_i(0)}(p_i, \frac{1}{2}) \times [-T, 0]$ are eventually uniformly bounded. By Theorem 2.7.3 the Ricci flows $g_i(t)$ are κ-non-collapsed. Hence $(C_{g_i(0)}(p_i, \frac{1}{2}), g_i(t), p_i)$, $t \in [-T, 0]$, subsequentially converges to a Ricci flow $(\mathcal{C}_\infty, g_\infty(t), p_\infty)$, $t \in [-T, 0]$, where by construction $b_s = 0$ at the point $(p_\infty, 0)$ in spacetime. Lemma 2.12.3 implies $Q = 1$ on $\mathcal{C}_\infty \times [-T, 0]$. Applying the strong maximum principle to the evolution equation (2.13.4) of b_s when $Q = 1$ shows that

$$b_s = 0 \text{ on } \mathcal{C}_\infty \times [-T, 0].$$

That is, the metric $g_\infty(t)$ is cylindrical. From here one can inductively show that for every $r > 0$ the Ricci flows $(C_{g_i(0)}(p_i, r), g_i(t), p_i)$, $t \in [-T, 0]$ subsequentially converge to a limiting cylindrical Ricci flow. Hence $(M_2, g_i(t), p_i)$, $t \in [-T, 0]$, subsequentially converges to the shrinking cylinder $(\mathbb{R} \times \mathbb{R}P^3, g_\infty(t), p_\infty)$, $t \in [-T, 0]$. As $T > 0$ is arbitrary the desired result follows by a diagonal argument. □

Before we carry on constructing the blow-up limit (iii) of Theorem 2.12.1, we need to state two technical lemmas. Their proofs can be skipped on the first reading.

Lemma 2.12.7. *Let $\hat{\eta} > 0$. There exists a $K = K(\hat{\eta}) > 1$ such that the following holds: Let $s_0 > 0$ and $b : [s_0, \infty) \to \mathbb{R}$ satisfy the ordinary differential inequalities*

$$bb_{ss} \leq b_s^2 - 1 + \left(1 - b_s^2\right)^2 \tag{2.12.2}$$

and

$$b_s > 0.$$

If at s_0 we have

$$\left.\frac{1 - b_s^2}{b^2}\right|_{s_0} = K \tag{2.12.3}$$

and

$$b_s\big|_{s_0} \in [\hat{\eta}, 1), \tag{2.12.4}$$

then $b_s < \hat{\eta}$ when $b \geq 1$.

Proof. First note that $b(s_0) \leq \frac{1}{\sqrt{K}} < 1$ by (2.12.3) and (2.12.4). Furthermore, as $b_s \in [\hat{\eta}, 1)$ at s_0 we see from (2.12.2) that $b_s < 1$ on $[s_0, \infty)$.

Write $B = b_s$. Then the ODI becomes

$$bB_s \leq B^2 - 1 + \left(1 - B^2\right)^2 = -B^2\left(1 - B^2\right).$$

Since $b_s > 0$ we may treat b as the independent variable, yielding the following ODI

$$\frac{\mathrm{d}B}{\mathrm{d}b} \leq -\frac{B^2\left(1 - B^2\right)}{bB}.$$

Note that as $B = b_s \in (0, 1)$ we may rearrange the inequality and integrate to obtain

$$\int_{B_0}^{B} \frac{B\mathrm{d}B}{B^2\left(1 - B^2\right)} \leq -\int_{b_0}^{b} \frac{\mathrm{d}b}{b},$$

where we denote by b_0 and B_0 the values of b and B at s_0, respectively. Evaluating the integrals and rearranging we deduce

$$\frac{B^2}{1 - B^2} \leq \frac{B_0^2 b_0^2}{(1 - B_0^2)b^2}.$$

By the initial conditions (2.12.3) and (2.12.4) we have

$$\frac{b_0^2 B_0^2}{1 - B_0^2} = \frac{B_0^2}{K} = \frac{B_0^4}{KB_0^2} \leq \frac{1}{K\hat{\eta}^2}$$

and therefore

$$\frac{B^2}{1-B^2} \leq \frac{1}{K\hat{\eta}^2 b^2}.$$

Hence when $b \geq 1$ we have

$$\frac{B^2}{1-B^2} \leq \frac{1}{K\hat{\eta}^2},$$

which can be rearranged to

$$B^2 \leq \frac{1}{K\hat{\eta}^2+1}.$$

Choosing K sufficiently large the desired result follows. $\square$

Now we may construct the orbifold Ricci flow blow-up:

Lemma 2.12.8. *Let $\eta \in (0,1)$ and $(M_2, g(t))$, $t \in [0, T_{sing})$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ with $\sup_{p\in M_2} b(p,0) < \infty$. Assume that (p_i, t_i) is a sequence of points in spacetime satisfying*

1. *$b(p_i, t_i) \to 0$ as $i \to \infty$*
2. *$\frac{b(p_i,t_i)}{b(o,t_i)} \to \infty$ as $i \to \infty$*
3. *$b_s(p_i, t_i) \to \eta$ as $i \to \infty$*

Consider the rescaled Ricci flows

$$g_i(t) = \frac{1}{b^2(p_i,t_i)} g(t_i + b^2(p_i,t_i)t), \quad t \in [-b(p_i,t_i)^{-2}t_i, 0].$$

Then $(M_2, g_i(t), p_i)$, $t \in [-b(p_i,t_i)^{-2}t_i, 0]$, subsequentially converges, in the Cheeger-Gromov sense, to an ancient Ricci flow $(M_\infty, g_\infty(t), p_\infty)$, $t \in (-\infty, 0]$, where $M_\infty \cong \mathbb{R}^4 \setminus \{0\}/\mathbb{Z}_2$. Moreover, $g_\infty(t)$ can be extended to a smooth orbifold Ricci flow on $\mathbb{R}^4/\mathbb{Z}_2$ that is homothetic to the 4d Bryant soliton quotiented by $\mathbb{Z}_2$.

Proof. Fix $T > 0$. By Lemma 2.12.5 we have that for every $\epsilon > 0$ there exists a $\delta > 0$ such that at points (p,t) in spacetime at which $b(p,t) < \delta$ we have $\partial_t b^2 \leq \epsilon$. This shows that for every $N > 0$ there exists a $\delta' = \delta'(N) > 0$ such that whenever $b(p,t) < \delta'$ then

$$b(p,t') > \frac{b(p,t)}{2} \text{ for } t' \in [t - Nb^2(p,t), t]. \tag{2.12.5}$$

Consider the rescaled metrics

$$g_i(t) = \frac{1}{b^2(p_i,t_i)} g\left(t_i + b^2(p_i,t_i)t\right).$$

For $n \in \mathbb{N}_{\geq 2}$ and $(p, t) \in M_2 \times [0, T_{sing})$ define the open set

$$E_{p,t,n} := \left\{ p' \in M_2 \,\middle|\, b(p', t) > \frac{b(p, t)}{n} \right\} \subseteq M_2$$

Furthermore, define the parabolic neighborhoods

$$\Omega_{i,n} = E_{p_i, t_i, n} \times [-T - 1, 0].$$

Recall that by Theorem 2.7.5 there exists a $C_1 > 0$ such that

$$|Rm_{g(t)}|_{g(t)} \leq \frac{C_1}{b^2} \;\text{ on }\; M_2 \times [0, T_{sing}).$$

Hence for fixed n and sufficiently large i the curvatures of $g_i(t)$ on $\Omega_{i,n}$ are uniformly bounded:

$$|Rm_{g_i(t)}|_{g_i(t)} \leq 4n^2 C_1 \;\text{ on } \Omega_{i,n}.$$

This follows from (2.12.5), $b(p_i, t_i) \to 0$ and the fact that $b_s \geq 0$ for metrics in $\mathcal{I}$. By Theorem 2.7.3 the Ricci flows $g_i(t)$ are κ-non-collapsed at larger and larger scales. By a slight adaptation of the local compactness Theorem 2.8.1 we see that for each $n \in \mathbb{N}$ the Ricci flows $(E_{p_i,t_i,n}, g_i(t), p_i)$, $t \in [-T-1, 0]$, subsequentially converge to a Ricci flow $(\mathcal{E}_{\infty,n}, g_{\infty,n}(t), p_{\infty,n})$, $t \in [-T-1, 0]$. The manifolds $\mathcal{E}_{\infty,n}$ are diffeomorphic to $\mathbb{R}^4 \setminus \{0\}/\mathbb{Z}_2$ and therefore incomplete. By a diagonal argument we may assume that $\mathcal{E}_{\infty,n} \subset \mathcal{E}_{\infty,n+1}$ and $g_{\infty,n}(t) = g_{\infty,n+1}(t)$ on $\mathcal{E}_{\infty,n}$. This allows us to drop the dependence on n and write $g_\infty(t)$ and p_∞ for brevity. By Lemma 2.12.3 we have $Q = 1$, $bb_{ss} \leq 0$, $T_{F_1} \leq 0$, $T_{F_2} \geq 0$ and $\partial_t b^2 \leq 0$ on $\mathcal{E}_{\infty,n}$.

Claim 1: There exists an $\hat{\eta} > 0$, independent of n, such that on the word line (p_∞, t), $t \in [-T, 0]$, in $\mathcal{E}_{\infty,n} \times [-T, 0]$ we have $b_s > \hat{\eta}$ uniformly.

Proof of Claim: We argue by contradiction. Assume that $t' \in [-T, 0]$ is such that for $t \in [t', 0]$ we have $b_s(p_\infty, t) \geq 0$ with equality if, and only if, $t = t'$. Applying the strong maximum principle to the evolution equation (2.13.4) of b_s when $Q = 1$, we obtain that $b_s = 0$ on $\mathcal{E}_{\infty,n} \times [-T-1, t']$. That is, the metric $g_\infty(t)$ is cylindrical for times $t \in [-T-1, t']$. We now show that this leads to a contradiction. Take times $t_i' = t_i - t' b^2(p_i, t_i)$. Then the spacetime points $(p_i, t_i') \in M_2 \times [0, T_{sing})$ converge to the spacetime point $(p_\infty, t') \in \mathcal{E}_{\infty,n} \times [-T-1, t']$. Consider the rescaled metrics

$$g_i'(t) = \frac{1}{b(p_i, t_i')^2} g(t_i' + t b(p_i, t_i')^2), \quad t \in [-t_i' b(p_i, t_i')^{-2}, 0].$$

Because $b_s(p_i, t_i') \to 0$ as $i \to \infty$, Lemma 2.12.6 implies that after passing to a subsequence $(M_2, g_i'(t), p_i)$ converges to the shrinking cylinder $\mathbb{R} \times \mathbb{R}P^3$. For every $n \in \mathbb{N}$ take $N_n \in \mathbb{N}$ such that for $i \geq N_n$ the region $C_{g_i'(0)}(p_i, n) \subset M_2$ is close, in the Cheeger-Gromov sense, to a cylinder $\mathbb{R} \times \mathbb{R}P^3$ of length $2n$ and radius 1. By Perelman's pseudolocality theorem there exists a $K > 0$ and $\tau > 0$ such that $g_i'(t)$ has bounded curvature on $C_{g_i'(0)}(p_i, n-1) \times [0, \tau]$. Hence $(M_2, g_i'(t), p_i)$, $t \in [-t_i' b(p_i, t_i')^{-2}, \tau]$, subsequentially converges to a limiting Ricci flow $(M_\infty, g_\infty'(t), p_\infty)$, $t \in (-\infty, \tau]$, where $M_\infty \cong \mathbb{R} \times \mathbb{R}P^3$ and $g_\infty'(0)$ is cylindrical. By the uniqueness of Ricci flow solutions [CZ06], we see that $g_\infty'(t)$ remains cylindrical for $t \in [0, \tau]$ and therefore $b_s = 0$ on $M_\infty \times (-\infty, \tau)$. Now we have arrived at a contradiction, as this implies that t' is not the earliest time at which $b_s = 0$ on the wordline through the point $(p_\infty, 0)$ in the spacetime $\mathcal{E}_{\infty,n} \times [-T-1, 0]$. This proves the claim. ∎

As $T_{F_1} \geq 0$ we have

$$-\frac{b_{ss}}{b} \leq \frac{1-b_s^2}{b^2} \ \text{ on } \ \mathcal{E}_{\infty,n} \times [-T, 0]$$

and hence

$$|Rm_{g_\infty(t)}|_{g_\infty(t)} \leq c\frac{1-b_s^2}{b^2} \tag{2.12.6}$$

for some universal constant $c > 0$, as $\frac{1-b_s^2}{b^2}$ and $-\frac{b_{ss}}{b}$ are the only non-zero curvature components of a rotationally symmetric metric.

Claim 2: There exists a $K = K(\hat{\eta}) > 1$, independent of n, such that

$$|Rm_{g_\infty(t)}|_{g_\infty(t)} < cK$$

uniformly on $\mathcal{E}_{\infty,n} \times [-T, 0]$.

Proof of Claim: Fix $n \geq 2$. As $bb_{ss} \leq 0$ and $b_s \geq 0$ it follows from Claim 1 that $b_s \geq \hat{\eta} > 0$ in the region

$$R = \left\{ (p,t) \in \mathcal{E}_{\infty,n} \times [-T, 0] \;\middle|\; b(p,t) \leq b(p_\infty, t) \right\}.$$

As $\partial_t b^2 \leq 0$ we have $b(p_\infty, t) \geq b(p_\infty, 0) = 1$ for $t \in [-T, 0]$.

Now choose a $K = K(\hat{\eta}) > 1$ such that Lemma 2.12.7 holds true. If at some point $(p', t') \in \mathcal{E}_{\infty,n} \times [-T, 0]$ we had

$$\frac{1-b_s^2}{b^2} \geq K$$

then on the time slice $\{t = t'\} \subset \mathcal{E}_{\infty,n}$ the result of Lemma 2.12.7 would imply that $b_s < \hat{\eta}$ when $b \geq 1$. This cannot be true, as by Lemma 2.12.5 we have that $\partial_t b^2 \leq 0$ on $\mathcal{E}_{\infty,n} \times [-T, 0]$ and therefore $b(p_\infty, t) \geq 1$ for $t \in [-T, 0]$. Hence we deduce by (2.12.6) that the curvature is bounded by cK on the region R. As on $\mathcal{E}_{\infty,n} \times [-T, 0] \setminus R$ we have $b > 1$, it follows by (2.12.6) and the fact that $b_s \in [0, 1]$ for metrics in $\mathcal{I}$ that the curvature is uniformly bounded by c there. ∎

Claim 2 shows that as $n \to \infty$ we may extract a limiting Ricci flow $(M_\infty, g_\infty(t), p_\infty)$, $t \in [-T, 0]$, with curvature bounded by cK. By construction M_∞ is diffeomorphic to $(R^4 \setminus \{0\})/\mathbb{Z}_2$. Define the radial coordinate $\xi : M_\infty \to \mathbb{R}$ by

$$\xi(p) = d_{g_\infty(0)}(p, \Sigma_{p_\infty}) + \xi_0,$$

where $\xi_0 \in \mathbb{R}$ is chosen such that $\xi \to 0$ as $b \to 0$.

Note that by the Ricci flow equation (2.2.12) for b we have

$$|\partial_t b^2| \leq 3b^2 |Rm_{g(t)}|_{g(t)} \leq 3b^2 K \text{ on } M_\infty \times [-T, 0].$$

Working in (ξ, t) coordinates we see that

$$b^2(\xi, t) \leq b^2(\xi, 0) e^{3KT}, \quad t \in [-T, 0].$$

Hence for all $t \in [-T, 0]$ we have $b(\xi, t) \to 0$ as $\xi \to 0$. As M_∞ has bounded curvature, we see that $\frac{1-b_s^2}{b^2}$ is bounded as well and hence $b_s(\xi, t) \to 1$ as $\xi \to 0$. From Theorem 2.14.1 in Appendix B it then follows that $g_\infty(t)$, $t \in (-T, 0]$, can be extended to a smooth orbifold Ricci flow on $\mathbb{R}^4 \times \mathbb{Z}_2$. Since T was arbitrary, a diagonal argument produces an ancient orbifold Ricci flow $(\mathbb{R}^4 \setminus /\mathbb{Z}_2, g_\infty(t), p_\infty)$, $t \in (-\infty, 0]$. Note that apriori $g_\infty(t)$ might have unbounded curvature as $t \to -\infty$.

As $Q = 1$, $b_s \in [0, 1]$ and $b_{ss} \leq 0$ we see that $g_\infty(t)$ is rotationally symmetric and has positive sectional curvature. Furthermore, for each $t \in (-\infty, 0]$ the metric $g_\infty(t)$ is asymptotically cylindrical, as the following argument shows: Either b is bounded, in which case $bb_{ss} \leq 0$ and $b_s \geq 0$ show that $\lim_{s\to\infty} b_s = 0$, or b is unbounded, in which case the inequality $T_{F_2} \leq 0$ and the proof of Lemma 2.12.7 show that on each time slice $b_s \to 0$ as $b \to \infty$.

By the Hamilton's trace Harnack inequality (see for instance [ChII, Theorem D.26]) and the fact that for any $T > 0$ the metric $g_\infty(t)$ has bounded curvature on $\mathbb{R}^4/\mathbb{Z}_2 \times [-T, 0]$, it follows that

$$\partial_t R_{g_\infty(t)} \geq 0 \text{ on } \mathbb{R}^4/\mathbb{Z}_2 \times (-\infty, 0].$$

Therefore $g_\infty(t)$ has bounded curvature on $\mathbb{R}^4/\mathbb{Z}_2 \times (\infty, 0]$. By the result of Li and Zhang [LZ18] we conclude that $g_\infty(t)$ is homothetic to the four dimensional Bryant soliton quotiented by $\mathbb{Z}_2$.

□

Lemma 2.12.9. *Let $(M_2, g(t))$, $t \in [0, T_{sing})$, be a Ricci flow starting from an initial metric $g(0) \in \mathcal{I}$ with $\sup_{p\in M_2} b(p, 0) < \infty$. Let (p_i, t_i) be a sequence of points in spacetime satisfying*

1. *$b(p_i, t_i) \to 0$*
2. *$b_s(p_i, t_i) \to 1$*

Consider the sequence of rescaled metrics

$$g_i(t) = \frac{1}{b^2(p_i, t_i)} g(t_i + b^2(p_i, t_i)t), \quad t \in [-b(p_i, t_i)^2 t_i, 0].$$

Then $(M_2, g_i(t), p_i)$, $t \in [-b(p_i, t_i)^2 t_i, 0]$, subsequentially converges, in the Cheeger-Gromov sense, to an ancient Ricci flow $(M_\infty, g_\infty(t), p_\infty)$, $t \in (-\infty, 0]$, where $M_\infty \cong \mathbb{R}^4 \setminus \{0\}/\mathbb{Z}_2$ and $g_\infty(t)$ can be extended to a smooth orbifold Ricci flow on $\mathbb{R}^4/\mathbb{Z}_2$ that is stationary and isometric to the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$.

Proof. First note that

Claim 1:

$$\frac{b(p_i,t_i)}{b(o,t_i)} \to \infty \text{ as } i \to \infty$$

Proof of Claim: We argue by contradiction. Assume there exists a $C > 0$ such that after passing to a subsequence of (p_i, t_i) we have

$$\frac{b(p_i,t_i)}{b(o,t_i)} \leq C.$$

Consider the rescaled metrics

$$g_i(t) = \frac{1}{b(p_i,t_i)^2} g\left(t_i + tb(p_i,t_i)^2\right), \quad t \in [-b(p_i,t_i)^{-2}t_i, 0].$$

Then by by Corollary 2.11.2 the sequence $(M_2, g_i(t), p_i)$ subsequentially converges to a blow-up limit $(M_2, g_\infty(t), p_\infty)$, which is homothetic to the Eguchi-Hanson space. By construction

$$b(p_\infty, 0) = 1$$

and

$$b_s(p_\infty, 0) = 1.$$

The latter follows from the assumption that $b_s(p_i, t_i) \to 1$ as $i \to \infty$. However, by Lemma 2.4.2 we have $b_s < 1$ everywhere on the Eguchi-Hanson space. This is a contradiction and the claim follows. ■

Fix $T > 0$ and consider the rescaled metrics $g_i(t)$ on the parabolic sets $E_{(p_i,t_i,n)} \times [-T, 0]$ as in the proof of Lemma 2.12.8. By the same reasoning, we see that for all $n \in \mathbb{N}_{\geq 2}$ the flows $(E_{p_i,t_i,n}, g_i(t), p_i)$ subsequentially converges to a Ricci flow $(\mathcal{E}_{\infty,n}, g_{\infty,n}(t), p_{\infty,n})$, $t \in [-T, 0]$. As in the proof of Lemma 2.12.8, we may assume that $\mathcal{E}_{\infty,n} \subset \mathcal{E}_{\infty,n+1}$ and $g_{\infty,n} = g_{\infty,n+1}$ on $\mathcal{E}_{\infty,n}$. Therefore we drop the dependence on n and write p_∞ and $g_\infty(t)$. By construction we have

$$b(p_\infty, 0) = 1$$

and

$$b_s(p_\infty, 0) = 1,$$

where the latter follows from the assumption that $b_s(p_i, t_i) \to 1$ as $i \to \infty$. Furthermore, by Lemma 2.12.3 and Claim 1 we have $Q = 1$ on $\mathcal{E}_{\infty,n} \times [-T, 0]$. Applying the strong maximum principle to the evolution equation (2.13.4) of b_s when $Q = 1$ we deduce that $b_s = 1$ everywhere in $\mathcal{E}_{\infty,n} \times [-T, 0]$. Hence $g_\infty(t)$ is flat and $(\mathcal{E}_{\infty,n}, g_\infty(t), p_\infty)$, $t \in [-T, 0]$, converges to the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$ as $n \to \infty$. As $T > 0$ was arbitrary the desired result follows by a diagonal argument. □

2.13 Appendix A: Evolution equations

Here we carry out some of the computations we rely on throughout Chapter 2. Recall

$$\frac{\partial}{\partial s} = \frac{1}{u(\xi,t)}\frac{\partial}{\partial \xi}$$

and the commutation relation

$$[\partial_t, \partial_s] = -\frac{a_{ss}}{a} - 2\frac{b_{ss}}{b}$$

from subsection 2.2. For the computations it will also be helpful to keep in mind that

$$bQ_s = a_s - Qb_s$$

which follows from differentiating the expression $Q = \frac{a}{b}$. Finally recall the definition of the Kähler quantities

$$x = a_s + Q^2 - 2$$

and

$$y = b_s - Q$$

from section 2.4.

First we compute the evolution equation of Q:

$$\partial_t Q = \frac{\partial_t a}{b} - \frac{a\partial_t b}{b^2}$$

Inserting the expressions for $\partial_t a$ and $\partial_t b$ from the evolution equations (2.2.11) and (2.2.12) for a and b we obtain

$$\partial_t Q = Q_{ss} + 3\frac{b_s}{b}Q_s + \frac{4}{b^2}Q(1-Q^2).$$

Evolution equations of a_s, b_s, Qb_s, x, y and $\frac{y}{Q}$

By the commutation relations above we have

$$\partial_t a_s = \partial_s\partial_t a - \left(\frac{a_{ss}}{a} + 2\frac{b_{ss}}{b}\right)a_s$$
$$\partial_t b_s = \partial_s\partial_t b - \left(\frac{a_{ss}}{a} + 2\frac{b_{ss}}{b}\right)b_s$$

Hence plugging in the expressions for $\partial_t a$ and $\partial_t b$ from the evolution equations (2.2.11) and (2.2.12) for a and b we obtain

$$\partial_t a_s = (a_s)_{ss} + \left(2\frac{b_s}{b} - \frac{a_s}{a}\right)(a_s)_s + \frac{1}{b^2}\left(-2a_s b_s^2 - 6Q^2 a_s + 8Q^3 b_s\right)$$
$$\partial_t b_s = (b_s)_{ss} + \frac{a_s}{a}(b_s)_s + \frac{1}{b^2}\left(-\frac{a_s^2 b_s}{Q^2} + 4Qa_s - 6Q^2 b_s - b_s^3 + 4b_s\right)$$

From here we can compute the evolution equation of Qb_s:

$$\begin{aligned}\partial_t Qb_s &= (\partial_t Q)b_s + Q\partial_t b_s \\ &= (Qb_s)_{ss} + \left(2\frac{b_s}{b} - \frac{a_s}{a}\right)(Qb_s)_s + \frac{1}{b^2}\left(4Q^2 a_s - 10Q^3 b_s - 2Qb_s^3 + 8Qb_s\right)\end{aligned}$$

Now we may compute the evolution equations of the Kähler quantities x and y:

$$\begin{aligned}\partial_t x &= \partial_t a_s + 2Q\partial_t Q \\ &= x_{ss} + \left(2\frac{b_s}{b} - \frac{a_s}{a}\right)x_s - \frac{2}{b^2}\left(2Q^2 + y^2\right)x - \frac{2}{b^2}\left(Q^2+2\right)y^2,\end{aligned}$$

where in the last step we made the substitutions $a_s = x - Q^2 + 2$, $b_s = y + Q$ and $a = Qb$. Similarly,

$$\begin{aligned}\partial_t y &= \partial_t b_s - \partial_t Q \\ &= y_{ss} + \frac{a_s}{a}y - \frac{y}{a^2}\left((x+2)^2 + Q^2\left(2x + y^2\right)\right)\end{aligned} \tag{2.13.1}$$

Then we can compute

$$\begin{aligned}\partial_t\left(\frac{y}{Q}\right) &= \frac{\partial_t y}{Q} - \frac{y\partial_t Q}{Q^2} \\ &= \left(\frac{y}{Q}\right)_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)\left(\frac{y}{Q}\right)_s + \frac{2}{b^2}\frac{y}{Q}\left(2 + \frac{y}{Q}\right)(Qb_s - 2a_s)\end{aligned} \tag{2.13.2}$$

where we substituted $a_s = x - Q^2 + 2$, $b_s = y + Q$ and $a = Qb$ in the last step.

Evolution equation of $H_\pm$

In section 2.7 we define the quantities

$$H_\pm := bb_{ss} \mp a_s^2 - b_s^2 \pm C$$

for some constant $C > 0$. Below we derive its evolution equation.

First note that we have

$$\partial_t b_{ss} = \partial_s\partial_t b_s - \left(\frac{a_{ss}}{a} + 2\frac{b_{ss}}{b}\right)b_s.$$

Substituting the evolution equation for b_s derived above we obtain

$$\begin{aligned}\partial_t b_{ss} = (b_{ss})_{ss} + \frac{a_s}{a}(b_{ss})_s + \frac{4aa_{ss}}{b^3} - \frac{2a_s^2 b_{ss}}{a^2} + \frac{2a_s^3 b_s}{a^3} \\ - \frac{24aa_s b_s}{b^4} + \frac{4a_s^2}{b^3} - \frac{2a_s a_{ss} b_s}{a^2} - \frac{6a^2 b_{ss}}{b^4} + \frac{24a^2 b_s^2}{b^5} \\ - \frac{2b_{ss}^2}{b} + \frac{4b_{ss}}{b^2} + \frac{2b_s^4}{b^3} - \frac{8b_s^2}{b^3} - \frac{3b_s^2 b_{ss}}{b^2}\end{aligned}$$

Hence we can compute the evolution equation of H via

$$\partial_t H = (\partial_t b)b_{ss} + b\partial_t b_{ss} \mp 2a_s\partial_t a_s - 2b_s\partial_t b_s$$

and substituting the expressions for $\partial_t b$, $\partial_t b_{ss}$, $\partial_t a_s$ and $\partial_t b_s$ derived above. Noting that

$$H_s = \mp 2a_s a_{ss} + b(b_{ss})_s - b_s b_{ss}$$

and

$$H_{ss} = \mp 2a_{ss}^2 \mp 2(a_{ss})_s a_s + b(b_{ss})_{ss} - b_{ss}^2$$

a longer computation shows that

$$\begin{aligned}\partial_t H_\pm = {} & [H_\pm]_{ss} + \left(\frac{a_s}{a} - 2\frac{b_s}{b}\right)[H_\pm]_s + H_\pm\left(-\frac{2a_s^2}{a^2} - \frac{4a^2}{b^4} - \frac{4b_s^2}{b^2}\right) \\ & \pm C\left(\frac{2a_s^2}{a^2} + \frac{4a^2}{b^4} + \frac{4b_s^2}{b^2}\right) \\ & \pm 2a_{ss}^2 + a_{ss}\left(-\frac{2ba_sb_s}{a^2} \mp \frac{8a_sb_s}{b} \pm \frac{4a_s^2}{a} + \frac{4a}{b^2}\right) \\ & + \frac{2ba_s^3b_s}{a^3} - \frac{32aa_sb_s}{b^3} \mp \frac{16a^3a_sb_s}{b^5} + \frac{4a_s^2}{b^2} \pm \frac{8a^2a_s^2}{b^4} \\ & \mp \frac{2a_s^4}{a^2} + \frac{32a^2b_s^2}{b^4} - \frac{16b_s^2}{b^2}.\end{aligned}$$

Evolution equation of $f_\theta(Q)$

$$\begin{aligned}\partial_t f_\theta(Q) &= f'\partial_t Q \\ &= f'\left(Q_{ss} + 3\frac{b_s}{b}Q_s + \frac{4}{b^2}Q\left(1 - Q^2\right)\right)\end{aligned}$$

by the evolution equation (2.5.1) of Q. Note that we omitted the dependence the quantities on spacetime (ξ, t) and and the dependence of f on Q and θ. For example we wrote f' for $f'_\theta(Q(\xi,t))$. Noting that

$$[f(Q)]_s = f'(Q)Q_s$$

and

$$[f(Q)]_{ss} = f''(Q)Q_s^2 + f'(Q)Q_{ss}$$

we obtain

$$\begin{aligned}
\partial_t f(Q) &= [f(Q)]_{ss} - f''Q_s^2 + 3\frac{b_s}{b}[f(Q)]_s + \frac{4}{b^2}f'Q\left(1-Q^2\right) \\
&= [f(Q)]_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)[f(Q)]_s \\
&\qquad + \left(5\frac{b_s}{b} - 3\frac{a_s}{a}\right)[f(Q)]_s + \frac{4}{b^2}f'Q\left(1-Q^2\right) - f''Q_s^2 \\
&= [f(Q)]_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)[f(Q)]_s + \frac{1}{b^2}C_f
\end{aligned}$$

where

$$\begin{aligned}
C_f &= \left(5b_s - 3\frac{a_s}{Q}\right)b[f(Q)]_s + 4f'Q\left(1-Q^2\right) - f''b^2Q_s^2 \\
&= \left(5b_s - 3\frac{a_s}{Q}\right)f'\left(a_s - Qb_s\right) + 4f'Q\left(1-Q^2\right) - \left(a_s - Qb_s\right)^2 f'' \\
&= \left(8a_sb_s - 3\frac{a_s^2}{Q} - 5Qb_s^2 + 4Q\left(1-Q^2\right)\right)f' - \left(a_s - Qb_s\right)^2 f''
\end{aligned}$$

Evolution equation of Z_θ

We have

$$\partial_t Z_\theta = \partial_t\left(\frac{x}{Q^2}\right) + \partial_t f_\theta(Q)$$

We computed the evolution equation for $f_\theta(Q)$ above. Therefore it remains to compute $\partial_t \frac{x}{Q^2}$. For this recall the evolution equation (2.5.7) of x

$$\partial_t x = x_{ss} + \left(2\frac{b_s}{b} - \frac{a_s}{a}\right)x_s + \frac{1}{b^2}C_x$$

where

$$C_x = -2\left(2Q^2 + y^2\right)x - 2\left(Q^2+2\right)y^2.$$

Differentiation shows that

$$\partial_s\left(\frac{x}{Q^2}\right) = \frac{x_s}{Q^2} - 2\frac{xQ_s}{Q^3}$$

and

$$\partial_{ss}\left(\frac{x}{Q^2}\right) = \frac{x_{ss}}{Q^2} - 4\frac{x_sQ_s}{Q^3} - 2x\frac{Q_{ss}}{Q^3} + 6x\frac{Q_s^2}{Q^4}.$$

Therefore we get

$$\begin{aligned}\partial_t \frac{x}{Q^2} &= \frac{1}{Q^2}\partial_t x - 2\frac{x}{Q^3}\partial_t Q \\ &= \left(\frac{x}{Q^2}\right)_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)\left(\frac{x}{Q^2}\right)_s + \frac{1}{b^2}C_{\frac{x}{Q^2}}\end{aligned}$$

where

$$\begin{aligned}C_{\frac{x}{Q^2}} &= 6\frac{xa_s}{Q^4}(bQ_s) - 10\frac{xb_s}{Q^3}(bQ_s) - 6\frac{x}{Q^4}(bQ_s)^2 + \frac{C_x}{Q^2} - \frac{8x}{Q^2}(1-Q^2) \\ &= -\frac{4a_s^2 b_s}{Q^3} + \frac{2a_s b_s^2}{Q^2} + \frac{8a_s b_s}{Q^3} - \frac{8a_s}{Q^2} + 2a_s - \frac{8b_s^2}{Q^2} + 8Qb_s + \frac{16}{Q^2} - 16\end{aligned}$$

In the last step, we used the expressions for x, y and Q_s in terms of a_s, b_s and Q to eliminate x, y and Q_s from the expression for $C_{\frac{x}{Q^2}}$. Hence we have

$$\partial_t Z_\theta = [Z_\theta]_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)[Z_\theta]_s + \frac{1}{b^2}C_Z$$

where

$$C_Z = C_{\frac{x}{Q^2}} + C_f.$$

As

$$Z_\theta = \frac{x}{Q^2} + f_\theta(Q) = \frac{a_s + Q^2 - 2}{Q^2} + f_\theta(Q)$$

by definition, we can solve for a_s to obtain

$$a_s = Q^2 Z_\theta - Q^2 f_\theta + 2 - Q^2.$$

Using this substitution to eliminate all occurring a_s from the expression C_Z we obtain

$$C_Z = C_{Z,0} + C_{Z,1}Z_\theta + C_{Z,2}Z_\theta^2$$

where

$$\begin{aligned}C_{Z,0} &= A_0 + A_1\left[\frac{b_s}{Q}\right] + A_2\left[\frac{b_s}{Q}\right]^2 \\ C_{Z,1} &= 2Q^3 b_s f'' + 8Q^2 b_s f' + 8fQb_s + 8Qb_s - \frac{8b_s}{Q} + 2b_s^2 + 2fQ^4 f'' \\ &\quad + 2Q^4 f'' - 4Q^2 f'' + 6fQ^3 f' + 6Q^3 f' - 12Qf' + 2Q^2 - 8 \\ C_{Z,2} &= -4Qb_s - Q^4 f'' - 3Q^3 f'\end{aligned}$$

and

$$\begin{aligned}
A_0 = & - Q^4 f^2 f'' - 2Q^4 f f'' - Q^4 f'' + 4Q^2 f f'' + 4Q^2 f'' - 4f'' - 3Q^3 f^2 f' \\
& - 6Q^3 f f' - 7Q^3 f' + 12Q f f' + 16Q f' - \frac{12 f'}{Q} - 2Q^2 f + 8f - 2Q^2 - 4 \\
A_1 = & - 2Q^4 f f'' - 2Q^4 f'' + 4Q^2 f'' - 8Q^3 f f' - 8Q^3 f' \\
& + 16Q f' - 4Q^2 f^2 - 8Q^2 f + 8f + 4Q^2 + 8 \\
A_2 = & - Q^4 f'' - 5Q^3 f' - 2Q^2 f - 2Q^2 - 4
\end{aligned}$$

Evolution equation of Z_1

The evolution equation for $Z_1 = \frac{x}{Q^2} + 1$ follows quickly from the evolution equations for Z_θ by setting $f = 1$. One obtains

$$\partial_t Z_1 = [Z_1]_{ss} + \left(3\frac{a_s}{a} - 2\frac{b_s}{b}\right)[Z_1]_s + \frac{1}{b^2}\left(C_{Z_1,0} + C_{Z_1,1} Z_1 + C_{Z_1,2} Z_1^2\right) \tag{2.13.3}$$

where

$$\begin{aligned}
C_{Z_1,0} &= \frac{1}{Q^2}\left(-4\left(1+Q^2\right)y^2 + 8Q\left(1-2Q^2\right)y + 16Q^2\left(1-Q^2\right)\right) \\
C_{Z_1,1} &= 16Q b_s - \frac{8b_s}{Q} + 2b_s^2 + 2Q^2 - 8 \\
C_{Z_1,2} &= -4Q b_s.
\end{aligned}$$

Note that we wrote $C_{Z_1,0}$ in terms of $y = b_s - Q$ instead of b_s in order to see the similarity with the zeroth order term of the evolution equation of T_1 presented in the proof of Lemma 2.5.8.

Evolution equations when $Q = 1$

When $Q = 1$ we have $a = b$ and the Ricci flow equations simplify. In particular, we obtain

$$\begin{aligned}
\frac{\partial_t u}{u} &= 3\frac{b_{ss}}{b} \\
\partial_t b &= b_{ss} + \frac{2}{b}\left(b_s^2 - 1\right)
\end{aligned}$$

Using the commutation relation of ∂_t and ∂_s we can also compute the evolution equation of b_s and b_{ss}:

$$\begin{aligned}
\partial_t b_s &= \partial_s \partial_t b - 3b_s \frac{b_{ss}}{b} \\
&= (b_s)_{ss} + \frac{b_s}{b}(b_s)_s + 2\frac{b_s}{b^2}\left(1 - b_s^2\right)
\end{aligned} \tag{2.13.4}$$

Similarly,

$$\begin{aligned}\partial_t b_{ss} &= \partial_s \partial_t b_s - 3b_{ss}\frac{b_{ss}}{b} \\ &= (b_{ss})_{ss} + \frac{b_s}{b}(b_{ss})_s - \frac{2b_{ss}^2}{b} + \frac{4\left(b_s^2-1\right)b_s^2}{b^3} \\ &\quad - \frac{5b_s^2 b_{ss}}{b^2} - \frac{2\left(b_s^2-1\right)b_{ss}}{b^2}\end{aligned} \tag{2.13.5}$$

Let us introduce the notation

$$\begin{aligned} X &:= 1 - b_s^2 \\ Y &:= -bb_{ss}. \end{aligned}$$

We need the evolution equations of *scale-invariant* quantities of the form

$$T_F = -Y + F(X),$$

where $F : [0,1] \to \mathbb{R}$ is a smooth function. In particular, we see that

$$\partial_t T_F = (\partial_t b) b_{ss} + b\partial_t b_{ss} - 2b_s F'\left(1 - b_s^2\right)\partial_t b_s$$

Expanding this expression, we obtain

$$\partial_t T_F = (T_F)_{ss} - \frac{b_s}{b}(T_F)_s + \frac{1}{b^2}C_F$$

where

$$\begin{aligned} C_F &= 4X^2 - 4XY - 4X - 2Y^2 + 4Y \\ &\quad + 2\left(2X^2 - 2XY - 2X + Y^2 + 2Y\right)F'(X) \\ &\quad + 4(X-1)Y^2 F''(X) \end{aligned}$$

In Chapter 2 we make use of three different choices of F:

$$\begin{aligned} F_0(X) &= 0 \\ F_1(X) &= X \\ F_2(X) &= X - X^2 \end{aligned}$$

Plugging these into the expression for C_F above we compute

$$\begin{aligned} C_{F_0} &= -4b_s^2\left(1 - b_s^2\right) - 2bb_{ss}\left(bb_{ss} + 2b_s^2\right) \\ C_{F_1} &= -8b_s^2 T_{F_1} \\ C_{F_2} &= 4\left((2-3X)Y^2 + \left(2X^2 - 4X + 2\right)Y - 2(X-1)^2 X\right) \end{aligned}$$

We also prove the following lemma here:

Lemma 2.13.1. *Let $X \in [0,1]$. Then whenever $0 \leq Y < X - X^2$ we have*

$$P(X,Y) \coloneqq (2-3X)Y^2 + \left(2X^2 - 4X + 2\right)Y - 2(X-1)^2X < 0.$$

In other words, $C_{F_2} < 0$ whenever $T_{F_2} > 0$ and $bb_{ss} \leq 0$.

Proof. Let R be the region in the X-Y-plane satisfying the inequalities $X \in [0,1]$ and $0 \leq Y < X - X^2$. Note that $Y < X - X^2$ and $Y \geq 0$ implies that $X \in (0,1)$. A computation shows

$$P(X, X - X^2) = -3(X-1)^2X^3 < 0 \;\text{ for }\; X \in (0,1)$$

Notice that for a fixed $X \in [0, \frac{2}{3}]$ the quadratic polynomial $P(X,Y)$ in Y is convex. As

$$P(X,0) = -2(X-1)^2X < 0 \;\text{ for }\; X \in (0,1)$$

we deduce that $P(X,Y) > 0$ on $R \cap \{X \leq \frac{2}{3}\}$. To prove that $P(X,Y) > 0$ in $R \cap \{X \geq \frac{2}{3}\}$ is trickier. For this we prove the following claim:

Claim 1: $\partial_X P(X,Y) > 0$ on $R \cap \{X \geq \frac{2}{3}\}$.

Proof of Claim: A computation shows

$$\partial_X P(X,Y) = -6X^2 + 8X - 2 + (4X-4)Y - 3Y^2.$$

Hence for fixed X is concave in Y. Then note that

$$\partial_X P(X,0) = -6X^2 + 8X - 2 > 0 \;\text{ for }\; X \in [\frac{2}{3}, 1)$$

and

$$\partial_X P(X, X - X^2) = (1-X)\left(3X^3 + X^2 + 2X - 2\right) > 0 \;\text{ for }\; X \in [\frac{2}{3}, 1).$$

The last inequality follows by the fact that the polynomial $3X^3 + X^2 + 2X - 2$ is increasing on $[0,1]$ and evaluates to $\frac{2}{3}$ at $X = \frac{2}{3}$. Hence the claim follows by concavity of $\partial_X P(X,Y)$ in Y. ■

By above we know that $P(X, X^2 - X) < 0$ for $X \in (0,1)$. Hence using the result of the claim, we see that $P(X,Y) < 0$ on $R \cap \{X \geq \frac{2}{3}\}$. □

2.14 Appendix B: Removable singularity

We prove the following theorem:

Theorem 2.14.1 (Removable singularity)**.** *Let $(\mathbb{R}^4 \setminus \{0\}, g(t))$, $t \in [0,T]$, be a rotationally symmetric Ricci flow of bounded curvature, i.e. there exists a $K > 0$ such that*

$$|Rm_{g(t)}|_{g(t)} < K \quad on \;\; \mathbb{R}^4 \times [0,T].$$

Taking $\xi \in (0,\infty)$ to be a radial coordinate on $\mathbb{R}^4 \setminus \{0\}$ the metric $g(t)$ may be written as

$$g(t) = u^2(\xi,t)d\xi^2 + b^2(\xi,t)g_{S^3},$$

where $u, b : (0,\infty) \to \mathbb{R}$ are smooth warping functions, and g_{S^3} is the round metric on S^3 with sectional curvatures equal to one. If for all $t \in [0,T]$ the warping function $b(\xi,t) \to 0$ as $\xi \to 0$, then $g(t)$ can be extended to a smooth Ricci flow on $\mathbb{R}^4 \times (0,T]$.

Below we assume $(\mathbb{R}^4 \setminus \{0\}, g(t))$, $t \in [0,T]$, is a Ricci flow as in Theorem 2.14.1. The proof strategy will be as follows: First we prove in Lemma 2.14.2 that for every $t_0 \in [0,T]$ there exist coordinates $x^i, i = 1,2,3,4$, of $\mathbb{R}^4$ for which the metric $g(t_0)$ can be extended to a $C^{1,1}$ metric on $\mathbb{R}^4$. Note, however, without control on the derivative of the curvature tensor the metric $g(t)$ at times $t \neq t_0$ may not to be $C^{1,1}$ with respect to the coordinates x^i. To get around this issue we show in Lemma 2.14.4 and Lemma 2.14.6 that in fact all derivatives $\nabla^m Rm$, $m \in \mathbb{N}$, of the curvature tensor are bounded on $\mathbb{R}^4 \setminus \{0\} \times (\delta, T]$ for any $\delta > 0$. The proof utilizes Shi's interior derivative estimates and is based on a De Giorgi-Nash-Moser iteration argument. With these results in place, we use harmonic coordinates to prove Theorem 2.14.1. Let us begin by proving

Lemma 2.14.2. *Let $g = ds^2 + b(s)^2 g_{S^3}$ be a smooth, rotationally symmetric metric with bounded curvature on $\mathbb{R}^4 \setminus \{0\}$. Here g_{S^3} is the round metric of curvature one on S^3 and $b : (0,\infty) \to \mathbb{R}$ is a smooth positive function. If*

$$b \to 0 \quad as \quad s \to 0$$

then g can be extended to a $C^{1,1}$ metric on $\mathbb{R}^4$. Furthermore, if we take standard Euclidean coordinates x_i, $i = 1,2,3,4$, on $\mathbb{R}^4$ we have $g_{ij} = \delta_{ij}$ and $\partial_k g_{ij} = 0$ at the origin, and $\partial_k \partial_l g_{ij}$ locally bounded on $\mathbb{R}^4 \setminus \{0\}$.

Proof. As g has bounded curvature there exists a $K > 0$ such that

$$\left|\frac{b_{ss}}{b}\right|, \left|\frac{1 - b_s^2}{b^2}\right| \leq K,$$

because these are the only non-zero curvature components of a rotationally symmetric metric. In particular, this shows that

$$b_s \to 1 \;\; \text{as} \;\; s \to 0^+$$

and

$$b_{ss} \to 0 \;\; \text{as} \;\; s \to 0^+.$$

Let x_i, $i = 1, 2, 3, 4$, be Euclidean coordinates of $\mathbb{R}^4$, normalized such that $\sum_i (x^i)^2 = s^2$. In these coordinates

$$g = \left[\delta_{ij} + \left(s^2\delta_{ij} - x_i x_j\right)\Psi(s)\right] \mathrm{d}x^i \mathrm{d}x^j,$$

where

$$\Psi(s) = \frac{\left(\frac{b}{s}\right)^2 - 1}{s^2}.$$

Note that we used the Einstein summation convention.

Claim 1: Ψ, $s\partial_s\Psi$, $s^2\partial_{ss}\Psi = O(K)$ as $s \to 0$.

Proof of Claim: Fix $s > 0$. By Taylor's theorem there exist numbers $s_0, s_1 \in (0, s)$ such that

$$\begin{aligned} b(s) &= s + \frac{1}{2} b_{ss}(s_0) s^2 \\ b_s(s) &= 1 + b_{ss}(s_1) s. \end{aligned}$$

Hence

$$\Psi(s) = \frac{b_{ss}(s_0)}{s} + \left(\frac{b_{ss}(s_0)}{2}\right)^2.$$

As $\left|\frac{b_{ss}}{b}\right| \leq K$, $b \to 0$ and $b_s \to 1$ as $s \to 0$, we see $\Psi(s) = O(K)$ as $s \to 0$. By similar reasoning one shows that

$$\begin{aligned} s\partial_s\Psi(s) &= \frac{2 - 4\left(\frac{b}{s}\right)^2 + 2\left(\frac{b}{s}\right) b_s}{s^2} \\ &= -\frac{3b_{ss}(s_0)}{s} + \frac{2b_{ss}(s_1)}{s} - b_{ss}(s_0)^2 + b_{ss}(s_1) b_{ss}(s_0) \end{aligned}$$

and

$$\begin{aligned} s^2\partial_{ss}\Psi(s) &= \frac{2bb_{ss} - 16\left(\frac{b}{s}\right) b_s + 2b_s^2 + 20\left(\frac{b}{s}\right)^2 - 6}{s^2} \\ &= b_{ss}(s_0) b_{ss}(s) + \frac{2b_{ss}(s)}{s} + \frac{12 b_{ss}(s_0)}{s} - \frac{12 b_{ss}(s_1)}{s} \\ &\qquad + 5b_{ss}(s_0)^2 - 8b_{ss}(s_1) b_{ss}(s_0) + 2b_{ss}(s_1)^2 \end{aligned}$$

are of order $O(K)$ as $s \to 0$. ■

Next, extend g to the origin by setting $g = \delta_{ij}$ there. As

$$\left(s^2\delta_{ij} - x_i x_j\right) = O(s^2)$$

it follows by Claim 1 that this defines a continuous extension of g to the origin.

A computation shows

$$\partial_k g_{ij} = (2x_k\delta_{ij} - \delta_{ik}x_j - x_i\delta_{jk})\,\Psi(s) + \left(s^2\delta_{ij} - x_ix_j\right)\frac{x_k}{s}\partial_s\Psi(s).$$

As

$$(2x_k\delta_{ij} - \delta_{ik}x_j - x_i\delta_{jk}) = O(s)$$

and

$$\left(s^2\delta_{ij} - x_ix_j\right)\frac{x_k}{s} = O(s^2),$$

it follows that we may continuously extend $\partial_k g_{ij}$ to the origin by setting $\partial_k g_{ij} = 0$ there. Finally, note that

$$\partial_k\partial_l g_{ij} = O(1)\Psi + O(s)\partial_s\Psi + O(s^2)\partial_{ss}\Psi = O(K).$$

Hence $\partial_k\partial_l g_{ij}$ is bounded in a neighborhood around, but excluding the origin. This shows that $\partial_k g_{ij}$ is Lipshitz. □

Next, we prove the boundedness of the gradient of the curvature tensor. For this we recall some interior curvature estimates. Note the following differential inequalities for the evolution of the curvature tensor and its derivatives under Ricci flow (See for instance [BC04, Chapter 7]):

$$\left(\partial_t - \Delta\right)|Rm|^2 \leq -2|\nabla Rm|^2 + 16|Rm|^3 \tag{2.14.1}$$

$$\left(\partial_t - \Delta\right)|\nabla^m Rm|^2 \leq -2|\nabla^{m+1} Rm|^2 + \sum_{j=0}^{m} c_{mj}|\nabla^j Rm|\cdot|\nabla^{m-j}Rm|\cdot|\nabla^m Rm| \tag{2.14.2}$$

Here c_{mj} are positive constants depending on j, m and the dimension of the manifold only. Note also that the laplacian is with respect to the evolving metric $g(t)$. Using these inequalities one can show the following interior derivative estimate (See for instance [CZ06, Theorem 1.4.2]).

Theorem 2.14.3 (Shi's interior estimates)**.** *There exist positive constants $\theta, C_m, m \in \mathbb{N}$, depending on the dimension n only, such that the following holds: Let M be a manifold of dimension n and $0 < T \leq \frac{\theta}{K}$. Assume that $g(t)$, $t \in [0,T]$, is a solution to the Ricci flow on an open neighborhood U of M and*

$$|Rm| < K \quad on \quad B_{g(0)}(p,r) \times [0,T].$$

If for $p \in U$ and $r > 0$ the closed set $\overline{B_{g(0)}(p,r)}$ is contained in U then

$$|\nabla^m Rm|^2 < C_m K^2\left(\frac{1}{r^{2m}} + \frac{1}{t^m} + K^m\right) \quad on \quad B_{g(0)}\left(p, \frac{r}{2}\right) \times (0,T]$$

Next, we prove that for all $\tau > 0$ the gradient $|\nabla Rm|$ is bounded on $\mathbb{R}^4 \setminus \{0\} \times [\tau, T]$. First note that due to Shi's estimates of Theorem 2.14.3

$$|\nabla Rm_{g(t)}|_{g(t)}(p) = O\left(\frac{1}{d_{g(t)}(p,0)}\right) \text{ on } \mathbb{R}^4 \setminus \{0\} \times [\tau, T].$$

Furthermore, from (2.14.2) and and an application of Kato's inequality to show that

$$|\nabla|\nabla Rm|| \leq |\nabla^2 Rm|$$

it follows that

$$\big(\partial_t - \Delta\big)|\nabla Rm| \leq C|Rm||\nabla Rm|.$$

Hence when curvature is bounded by K, the function $u := e^{-CKt}|\nabla Rm|$ is a subsolution to the heat equation, i.e.

$$\big(\partial_t - \Delta\big)u \leq 0.$$

With help of a De Giorgi-Nash-Moser iteration argument, this is enough to prove that u is bounded for $t > \tau$. We carry this out in the lemma below:

Lemma 2.14.4. *Let $(\mathbb{R}^4 \setminus \{0\}, g(t))$, $t \in [0,T]$, be a Ricci flow as in Theorem 2.14.1. Then for any $\tau > 0$ there exists a $C = C(K,\tau) > 0$ such that*

$$|\nabla Rm| < C$$

on $\mathbb{R}^4 \times \setminus \{0\} \times [\tau, T]$.

Proof. As shown above, the function $u = e^{-CKt}|\nabla Rm|$ is a subsolution to the heat equation, i.e.

$$(\partial_t - \Delta)\, u \leq 0.$$

By Lemma (2.14.2) we may choose Euclidean coordinates x^i, $i = 1,2,3,4$, on $\mathbb{R}^4$ for which $g(0)$ is $C^{1,1}$. Take $s^2 = \sum_i (x^i)^2$ and write $B_R(x)$ for the ball centered at x with radius R with respect to $g(0)$.

Since the curvature of $g(t)$ is bounded on $\mathbb{R}^4 \setminus \{0\} \times [0,T]$, there exists a $\lambda > 0$ such that

$$\frac{1}{\lambda} g(0) \leq g(t) \leq \lambda g(0) \text{ on } \mathbb{R}^4 \setminus \{0\} \times [0,T].$$

Therefore, Shi's interior estimates imply

$$u(\cdot, t) = O\left(\frac{1}{s}\right) \text{ for } t \in [\tau, T].$$

Hence it suffices to show that for some $R > 0$ the function u is bounded on $B_R(0) \times [\tau, T]$. We achieve this via a De Giorgi-Nash-Moser iteration argument. In the Claim below we derive the crucial estimate.

Claim 1: Let $\delta > 0$, $p \geq 2$, $R_0 \in [1, 10]$ and $t_0 \in [\delta, T)$. Then there exists a constant $C = C(K, \delta, T) > 0$ such that the following holds: If $u \in L^p(B_{R_0} \times [t_0, T])$, then for $R_1 \in [\frac{1}{2}, R_0)$ and $t_1 \in (t_0, T]$

$$\|u\|_{L^{2p}(B_{R_1}(0)\times[t_1,T])} \leq \left[C \left(\frac{p^2}{(R_0 - R_1)^2} + \frac{1}{t_1 - t_0} \right) \right]^{\frac{1}{p}} \|u\|_{L^p(B_{R_0}(0)\times[t_0,T])}.$$

Proof of Claim: In the following a constant C is assumed to only depend on K, δ and T and might vary from line to line. Fix a number $A > 1$ that we later take to ∞. Then choose a C^2 function $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with the following properties:

1. $F(s) = s^p$ for $s \leq A$
2. F is linear for $s \geq A + 1$ with slope $pA^{p-1} + 1$
3. On $[A, A + 1]$ take $F(s)$ to be defined such that $F'' \geq 0$

We see that these properties imply that $F'(s) \leq ps^{p-1}$. Next, define the cut-off functions η_ϵ, $\epsilon > 0$, and $\phi : \mathbb{R}^4 \to \mathbb{R}$. For this take a smooth function $h : \mathbb{R} \to \mathbb{R}$ with $h = 0$ on $(-\infty, \frac{1}{2}]$ and $h = 1$ on $[1, \infty)$. Then define

$$\eta_\epsilon = h\left(\frac{s}{\epsilon}\right)$$

and

$$\phi = h\left(\frac{R_0 - s}{R_0 - R_1}\right).$$

That is, $\phi = 1$ on $B_{R_1}(0)$ and $\phi = 0$ on $\mathbb{R}^4 \setminus B_{R_0}(0)$. Clearly, $|\nabla\phi|_{g(t)} \leq \frac{c}{R_0 - R_1}$ and $|\nabla\eta_\epsilon|_{g(t)} \leq \frac{c}{\epsilon}$ for some universal constant c depending on h and λ only.

Since $u \in L^p(B_{R_0} \times [t_0, T])$ is a positive function there exists a $t' \in [t_0, t_1]$ such that

$$\int_{B_{R_0}(0)} u^p(\cdot, t')\, \mathrm{d}x \leq \frac{1}{t_0 - t_1} \int_{t_0}^{T} \int_{B_{R_0}(0)} u^p\, \mathrm{d}x\, \mathrm{d}t. \tag{2.14.3}$$

Next, we compute via integration by parts

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^4} \eta_\epsilon F(u)\phi^2\, \mathrm{d}x &= \int_{\mathbb{R}^4} \eta_\epsilon F'(u)\Delta u\phi^2\, \mathrm{d}x \\ &= -\int_{\mathbb{R}^4} \nabla\eta_\epsilon F'(u)\nabla u\phi^2\, \mathrm{d}x - \int_{\mathbb{R}^4} \eta_\epsilon F''(u)|\nabla u|^2\phi^2\, \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^4} \eta_\epsilon F'(u)\nabla u\nabla\phi^2\, \mathrm{d}x. \end{aligned}$$

Integrating with respect to time from t' to T and noting that

$$\int_{\mathbb{R}^4} \eta_\epsilon F(u(\cdot, T))\phi^2\, \mathrm{d}x \geq 0,$$

we obtain

$$\int_{t'}^{T}\int_{\mathbb{R}^4}\eta_\epsilon F''(u)|\nabla u|^2\phi^2\,\mathrm{d}x\,\mathrm{d}t \le \int_{\mathbb{R}^4}\eta_\epsilon F(u(\cdot,t'))\phi^2\,\mathrm{d}x - \int_{t'}^{T}\int_{\mathbb{R}^4}\nabla\eta_\epsilon F'(u)\nabla u\phi^2\,\mathrm{d}x\,\mathrm{d}t \tag{2.14.4}$$
$$- \int_{t'}^{T}\int_{\mathbb{R}^4}\eta_\epsilon F'(u)\nabla u\nabla\phi^2\,\mathrm{d}x\,\mathrm{d}t$$
$$:= I_1 - I_2 - I_3.$$

Next we estimate each of these integrals I_1, I_2 and I_3 separately, in order to analyze their behaviors as $\epsilon \to 0$. For the first integral we have

$$I_1 \le \int_{B_{R_0}} u^p(\cdot,t')\,\mathrm{d}x \le \frac{1}{t_1 - t_0}\,\|u\|^p_{L^p(B_{R_0}(0)\times[t_0,T])}\,.$$

For the second integral I_2, note that Shi's estimates and Kato's inequality yield

$$|\nabla u| = |\nabla\left(e^{-CKt}|\nabla Rm|\right)| \le |\nabla^2 Rm| = O\left(\frac{1}{s^2}\right) \quad \text{as } s \to 0.$$

As $|\nabla\eta_\epsilon| \le \frac{c}{\epsilon}$, $|F'| \le pA^{p-1}+1$ and $\phi^2 = 1$ in a neighborhood of 0, we see that

$$|\nabla\eta_\epsilon|\cdot|F'(u)|\cdot|\nabla u|\cdot|\phi^2| \le C\epsilon^{-3}(pA^{p-1}+1) \ \text{ on } \ B_\epsilon(0).$$

As $\mathrm{vol}_{g(t)}(B_\epsilon(0)) \le C\epsilon^4$ we obtain

$$|I_2| \le C\epsilon(pA^{p-1}+1).$$

For the final integral I_3, recall that by definition $0 \le F'(u) \le pu^{p-1}$. Furthermore, $|\nabla\phi^2|$ has support in $B_{R_0}(0)\setminus B_{R_1}(0)$ and is bounded by $\frac{2c}{R_0-R_1}$. As $R_1 \ge \frac{1}{2}$ by assumption, Shi's estimates imply that on this set $|\nabla u|$ and u are bounded by some constant C. Thus

$$|I_3| \le \frac{2cpC^p}{R_0-R_1}\mathrm{vol}\left(B_{R_0}(0)\setminus B_{R_1}(0)\right)$$
$$\le pC^{p+1},$$

where we used that

$$\mathrm{vol}(B_{R_0}(0)\setminus B_{R_1}(0)) \le C\left(R_0^4 - R_1^4\right) \le C\left(R_0 - R_1\right),$$

as $\frac{1}{2} \le R_0 \le R_1 \le 10$ by assumption. This shows that I_3 is convergent. Now split the integral I_3 as

$$I_3 = \int_{t'}^{T}\int_{\{u\le A\}}\eta_\epsilon F'(u)\nabla u\nabla\phi^2\,\mathrm{d}x\,\mathrm{d}t + I_4,$$

where

$$I_4 = \int_{t'}^{T} \int_{\{u \geq A\}} \eta_\epsilon F'(u) \nabla u \nabla \phi^2 \text{ d}x \text{ d}t.$$

As I_3 is convergent, we see that $I_4 \to 0$ as $A \to \infty$. Observe that by Young's inequality

$$pu^{p-1}|\nabla u||\nabla \phi^2| = \phi u^{\frac{p-2}{2}}|\nabla u| \cdot 2pu^{\frac{p}{2}}|\nabla \phi| \leq \frac{1}{2}u^{p-2}|\nabla u|^2\phi^2 + 2p^2|\nabla \phi|^2 u^p.$$

Moreover,

$$|\nabla \phi|^2 \leq \left(\frac{c}{R_0 - R_1}\right)^2.$$

Hence we obtain

$$\begin{aligned} |I_3| &\leq \int_{t'}^{T} \int_{\{u \leq A\}} \frac{1}{2}\eta_\epsilon u^{p-2}|\nabla u|^2\phi^2 + 2p^2\eta_\epsilon|\nabla\phi|^2 u^p \text{ d}x \text{ d}t + I_4 \\ &\leq \int_{t'}^{T} \int_{\{u \leq A\}} \frac{1}{2}\eta_\epsilon u^{p-2}|\nabla u|^2\phi^2 \text{ d}x \text{ d}t + \frac{Cp^2}{(R_0-R_1)^2}\|u\|^p_{L^p(B_{R_0}(0)\times[t_0,T])} + I_4. \end{aligned}$$

Next, note that $F''(u) = p(p-1)u^{p-2}$ for $u \leq A$ and $F'' \geq 0$ everywhere. Therefore

$$\int_{t'}^{T} \int_{\mathbb{R}^4} \eta_\epsilon F''(u)|\nabla u|^2\phi^2 \text{ d}x \text{ d}t \geq \int_{t'}^{T} \int_{\{u \leq A\}} \eta_\epsilon p(p-1)u^{p-2}|\nabla u|^2\phi^2 \text{ d}x \text{ d}t.$$

Substituting this inequality and the inequalities for $|I_1|$, $|I_2|$ and $|I_3|$ derived above into (2.14.4), we deduce

$$\begin{aligned} &\left(p(p-1) - \frac{1}{2}\right)\int_{t'}^{T} \int_{\{u \leq A\}} \eta_\epsilon u^{p-2}|\nabla u|^2\phi^2 \text{ d}x \text{ d}t \\ &\qquad \leq C\left(\frac{p^2}{(R_0-R_1)^2} + \frac{1}{t_1 - t_0}\right)\|u\|^p_{L^p(B_{R_0}(0)\times[t_0,T])} + C\epsilon(pA^{p-1}+1) + I_4 \end{aligned}$$

Taking $\epsilon \to 0$ and then $A \to \infty$ yields

$$\begin{aligned} &\left(p(p-1) - \frac{1}{2}\right)\int_{t'}^{T} \int_{\mathbb{R}^4} u^{p-2}|\nabla u|^2\phi^2 \text{ d}x \text{ d}t \\ &\qquad \leq C\left(\frac{p^2}{(R_0-R_1)^2} + \frac{1}{t_1 - t_0}\right)\|u\|^p_{L^p(B_{R_0}(0)\times[t_0,T])} \end{aligned}$$

by the monotone convergence theorem. Then note that

$$u^{p-2}|\nabla u|^2\phi^2 = \frac{4}{p^2}|\nabla u^{\frac{p}{2}}|^2\phi^2$$

and

$$\begin{aligned}|\nabla u^{\frac{p}{2}}|^2\phi^2 &= \left|\nabla(\phi u^{\frac{p}{2}}) - u^{\frac{p}{2}}\nabla\phi\right|^2 \\ &\geq \left|\nabla(\phi u^{\frac{p}{2}})\right|^2 + \left|u^{\frac{p}{2}}\nabla\phi\right|^2 - 2\left|\nabla(\phi u^{\frac{p}{2}})\right| \cdot \left|u^{\frac{p}{2}}\nabla\phi\right| \\ &\geq \frac{1}{2}|\nabla(\phi u^{\frac{p}{2}})|^2 - u^p|\nabla\phi|^2,\end{aligned}$$

where in the last line we applied Young's inequality to bound the cross-term. Therefore

$$\int_{t'}^{T}\int_{\mathbb{R}^4} |\nabla(\phi u^{\frac{p}{2}})|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C\left(\frac{p^2}{(R_0-R_1)^2} + \frac{1}{t_1-t_0}\right)\|u\|^p_{L^p(B_{R_0}(0)\times[t_0,T])}$$

and applying the Sobolev inequality proves Claim 1. ■

Now we may iterate the estimate of Claim 1 to prove the desired result. First note that due to Shi's estimates, for any $R_0 > 0$ and $t_0 > 0$ we have that $u \in L^2(B_{R_0}(0) \times [t_0, T])$. We take $t_0 = \frac{\tau}{2}$, $R_0 = 2 + \sqrt{\frac{\tau}{2}}$, $\Delta t_i = (\Delta R_i)^2 = \frac{\tau}{2^{i+1}}$, $p_i = 2^{i+1}$ and

$$\begin{aligned}R_{i+1} &= R_i - \Delta R_i \\ t_{i+1} &= t_i + \Delta t_i.\end{aligned}$$

Then inductively applying the estimate of Claim 1 and taking the limit as $i \to \infty$, we obtain

$$\|u\|_{L^\infty(B_2(0)\times[\tau,T])} \leq C_\infty \|u\|_{L^2(B_{R_0}(0)\times[\frac{\tau}{2},T])} < \infty,$$

where $C_\infty > 0$ is a positive constant. This proves the desired result.

□

Next, we prove that the higher derivatives of the curvature tensor are also bounded at positive times. For this we need a generalization of Shi's estimates for the situation in which some of the derivatives of the curvature tensor are known to be bounded. In particular, we have

Theorem 2.14.5 (Shi's interior estimates with derivative bounds)**.** *Let $n \geq 2$ and $m \geq 1$. Then for every choice of constant $K > 0$ there exists constants $\theta > 0$ and $C > 0$ such that the following holds: Let M be an open manifold M of dimension n and $0 < T \leq \frac{\theta}{K}$. Assume that $g(t)$, $t \in [0, T]$, is a solution to the Ricci flow on an open subset U of M and*

$$|\nabla^l Rm| \leq K \quad \textit{on} \;\; U \times [0,T] \;\; \textit{and for} \;\; l \in \{0, 1, 2, \cdots, m\}$$

If for $p \in U$ and $r > 0$ the closed set $\overline{B_{g(0)}(p,r)}$ is contained in U then

$$|\nabla^{m+1}Rm|^2 \leq C\left(\frac{1}{r^2} + \frac{1}{t} + 1\right) \quad \textit{on} \;\; B_{g(0)}\left(p, \frac{r}{2}\right) \times (0,T]$$

Proof. We follow the proofs of [CZ06, Theorem 1.4.2] and [ChII, Theorem 14.16]. In the following the constant C depends on m and n only and may vary from line to line. Consider the quantity

$$S = \left(BK^2 + |\nabla^m Rm|^2\right)|\nabla^{m+1}Rm|^2,$$

where $B > 0$ is to be fixed later. With help of the differential inequality (2.14.2) we obtain

$$\begin{aligned}
\partial_t S \le\ & \Delta S - 2\nabla|\nabla^m Rm|^2 \nabla|\nabla^{m+1}Rm|^2 - 2|\nabla^{m+1}Rm|^4 \\
& + \sum_j c_{mj} \cdot |\nabla^j Rm| \cdot |\nabla^{m-j}Rm| \cdot |\nabla^m Rm| \cdot |\nabla^{m+1}Rm|^2 \\
& - 2\left(BK^2 + |\nabla^m Rm|^2\right)|\nabla^{m+2}Rm|^2 \\
& + \left(BK^2 + |\nabla^m Rm|^2\right)\sum_j c_{m+1j} \cdot |\nabla^j Rm| \cdot |\nabla^{m+1-j}Rm| \cdot |\nabla^{m+1}Rm|
\end{aligned}$$

Using Cauchy's inequality and the assumption that $|\nabla^l Rm| \le K$ for $l = 0, 1, 2, \cdots m$, we deduce

$$\begin{aligned}
\partial_t S \le\ & \Delta S + 8K|\nabla^{m+1}Rm|^2|\nabla^{m+2}Rm| - 2|\nabla^{m+1}Rm|^4 - 2BK^2|\nabla^{m+2}Rm|^2 \\
& + CK^3|\nabla^{m+1}Rm|^2 + CK^3(B+1)\left(|\nabla^{m+1}Rm|^2 + K|\nabla^{m+1}Rm|\right)
\end{aligned}$$

Noting that for all $x \in \mathbb{R}$ we have $x^2 + Kx \le 2x^2 + \frac{1}{4}K^2$ we obtain with help of Young's inequality that

$$\begin{aligned}
\partial_t S \le\ & \Delta S - |\nabla^{m+1}Rm|^4 + 2\left(32 - B\right)K^2|\nabla^{m+2}Rm|^2 \\
& + CK^6 + CK^5(B+1) + CK^6(B+1)^2.
\end{aligned}$$

Taking $B = 32$ and assuming without loss of generality that $K > 1$, we obtain

$$\partial_t S \le \Delta S - \frac{S^2}{CK^4} + CK^6$$

From here we may follow the proof of [CZ06, Theorem 1.4.2] to deduce the desired result. $\square$

With help of Theorem 2.14.5 we inductively prove that the higher derivatives of the curvature tensor are bounded.

Lemma 2.14.6. *Let $(\mathbb{R}^4 \setminus \{0\}, g(t))$, $t \in [0, T]$, be a Ricci flow as in Theorem 2.14.1. Then for any $\tau > 0$ there exist constants $C_m = C_m(K, \tau) > 0$, $m \in \mathbb{N}$, such that*

$$|\nabla^m Rm| < C_m$$

on $\mathbb{R}^4 \times \setminus\{0\} \times [\tau, T]$.

Proof. We prove this lemma by induction. By Lemma 2.14.4 the result is true for $m = 1$. Assume that the result is true for $m \leq N$. Then there exist constants $C_l > 0$, $l = 1, 2, 3, \cdots, N$ such that

$$|\nabla^l Rm| \leq C_l \ \text{ on } \ \mathbb{R}^4 \setminus \{0\} \times \left[\frac{\tau}{4}, T\right] \ \text{ and for } \ l = 1, 2, 3, \cdots, N.$$

As in the proof of Lemma 2.14.4, choose coordinates x^i, $i = 1, 2, 3, 4$, such that $g(0)$ can be extended to a $C^{1,1}$ metric on $\mathbb{R}^4$, and write $s^2 = \sum_i (x^i)^2$. As the curvature of $g(t)$, $t \in [0, T]$, is bounded there exists a $\lambda > 0$ such that

$$\frac{1}{\lambda} g(0) \leq g(t) \leq \lambda g(0) \ \text{ on } \ \mathbb{R}^4 \setminus \{0\} \times [0, T].$$

By the modified Shi's estimates of Theorem 2.14.5 we see that

$$|\nabla^{N+1} Rm| \leq C \left(\frac{1}{s} + 1\right) \ \text{ on } \ \mathbb{R}^4 \setminus \{0\} \times \left[\frac{\tau}{2}, T\right],$$

for some C that depends on τ, K, and C_l, $l = 1, 2, \cdots, N$, only. In particular, this implies that for all $R > 0$ the function $u \in L^2(B_R(0)) \times [0, T]$. By the differential inequality (2.14.2) for the evolution of the curvature derivatives we see that

$$(\partial_t - \Delta) |\nabla^{N+1} Rm|^2 \leq -2|\nabla^{N+2} Rm|^2 + CK^2 |\nabla^{N+1} Rm| + CK |\nabla^{N+1} Rm|^2$$

and hence

$$(\partial_t - \Delta) |\nabla^{N+1} Rm| \leq CK \left(K + |\nabla^{N+1} Rm|\right).$$

Thus defining

$$u = e^{-CKt} \left(|\nabla^{N+1} Rm| + K\right)$$

we deduce that

$$(\partial_t - \Delta) u \leq 0.$$

Now we are in the same setup as in the proof of Lemma 2.14.4. Therefore we may use the same De Giorgi-Nash-Moser iteration argument to show that u and hence $|\nabla^{N+1} Rm|$ are bounded in $\mathbb{R}^4 \times [\tau, T]$. This proves the desired result. $\square$

Now we may prove the main Theorem 2.14.1:

Proof of Theorem 2.14.1. By Lemma 2.14.2 we can choose coordinates x^i, $i = 1, 2, 3, 4$, for $\mathbb{R}^4$ such that $g(T)$ can be extended to a $C^{1,1}$ metric on all of $\mathbb{R}^4$. Below we write $g = g(T)$ for brevity. By [DK81, Lemma 1.2] there exist $C^{2,\alpha}$ harmonic coordinates $y^i : U \to \mathbb{R}$, $i = 1, 2, 3, 4$, in an open neighborhood U of $\mathbb{R}^4$ containing the origin and satisfying

1. $y^i = 0$
2. $\frac{\partial y^i}{\partial x^j} = \delta^i_j$

at the origin. Furthermore, as g is smooth on $U\setminus\{0\}$, it follows from interior elliptic regularity that y^i are smooth on $U \setminus \{0\}$. Write

$$g_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) \text{ and } Ric_{ij} = Ric_g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right).$$

We have that $g_{ij} \in C^{1,\alpha}(U)$ with respect to the y^i coordinates. Furthermore, g_{ij} is smooth on $U \setminus \{0\}$.

By [PP, Chapter 10, Lemma 49]) we have

$$\frac{1}{2}\Delta g_{ij} + Q(g, \partial g) = -Ric_{ij} \text{ on } U \setminus \{0\}, \tag{2.14.5}$$

where $Q(g, \partial g)$ is some universal analytic expression that is polynomial in the matrix g, quadratic in $\frac{\partial g}{\partial y^i}$, and has a denominator term depending on $\sqrt{det\, g_{ij}}$. The equation (2.14.5) makes sense on all of U if we interprete it in the weak sense.

Claim 1: If $g_{ij}(y) \in C^k(U)$ for $k \in \mathbb{N}$ then $R_{ij}(y) \in C^{k-1,1}(U)$.

Proof of Claim: Write

$$Y_i = \frac{\partial}{\partial y^i} \text{ for } i = 1, 2, 3, 4.$$

on $U \setminus \{0\}$ we have

$$\begin{aligned}\frac{\partial^k}{\partial y^{i_1}\partial y^{i_2}\cdots\partial y^{i_k}}Ric_{ij} &= Y_{i_1}Y_{i_2}\cdots Y_{i_k}Ric(Y_i, Y_j)\\ &= \nabla_{Y_{i_1}}\nabla_{Y_{i_2}}\cdots\nabla_{Y_{i_k}}Ric(Y_i, Y_j).\end{aligned}$$

Since covariant differentiation commutes with contractions, we can use the product rule to express the above derivative as a sum of terms, which only involve $\nabla^m Ric$, $m = 1, 2, \cdots, k$, and $\nabla^m Y_{i_l}$, $m, l = 1, 2, 3, \cdots, k$, contracted with Y_i, Y_j and Y_{i_l}, $l = 1, 2, \cdots, k$. As by Lemma 2.14.6 all the derivatives of the curvature tensor are bounded and $g_{ij}(y) \in C^k(U)$ we see that

$$\frac{\partial^k}{\partial y^{i_1}\partial y^{i_2}\cdots\partial y^{i_k}}Ric_{ij}$$

is bounded as well. Hence the k-th spatial derivatives $\partial^k Ric_{ij}$ are bounded, which implies that $\partial^{k-1}Ric_{ij}$ is a Lipshitz function and can be continuously extended to a all of U. Similarly, the lower order derivatives $\partial^m Ric_{ij}$, $m = 0, 1, 2, \cdots, k-2$, can be continuously extended to the origin. ∎

First note that g is a $C^{1,1}(U)$ weak solution of the elliptic equation (2.14.5). Furthermore $Q(g, \partial g) \in C^{0,1}(U)$ and by Claim 1 we have $Ric_{ij} \in C^{0,1}(U)$ as well. Since such weak solutions are unique, and there exists a $C^{2,\alpha}(U)$ solution that agrees on the boundary ∂U, we see that g is in fact $C^{2,\alpha}(U)$. Bootstrapping standard Schauder estimates and the result of Claim 1, we conclude that g_{ij} is smooth with respect to the harmonic coordinates y^i, $i = 1, 2, 3, 4$.

It remains to be shown that $g(t)$ can be extended to a smooth Ricci flow on $\mathbb{R}^4 \times (0, T]$. Recall that by Lemma 2.14.6, for all $\tau > 0$ the derivatives of the curvature tensor are bounded on $U \times [\tau, T]$. Moreover $g(T)$ is bi-lipshitz to the euclidean metric δ_{ij} on U and by the previous paragraph the covariant derivatives of $g(T)$ with respect to δ_{ij} are all bounded. Therefore we may follow the proof of [ChI, Lemma 3.11] with $t_0 = T$ to deduce that

$$\frac{\partial^m}{\partial t^m} \frac{\partial^n}{\partial y^{i_1} \cdots \partial y^{i_n}} (g(t))_{ij} \leq K_{m,n} \text{ on } U \setminus \{0\} \times [\tau, T],$$

for some constants $K_{m,n} > 0$. This shows that $g(t)$ can be smoothly extended to $U \times [\tau, T]$. As $\tau > 0$ was arbitrary the desired result follows. $\square$

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